The governing equations for CM1

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1 Introduction

This document describes the governing equations for the CM1 numerical model, valid for release 18 (cm1r18, released in August 2015). This document is being provided because changes have occurred since the original release of the model (cm1r01, January 2003), which was described by Bryan (2002) and Bryan and Fritsch (2002). Differences in the governing equations between cm1r01 and cm1r18 are mostly minor; however, this document is being provided for clarity and also to provide details that are not available in the two articles cited above.

Definitions of many symbols are provided in Table 1 (for variables that are arrays in the code) and Table 2 (for variables that are constants in the code).

2 Governing equations

The model integrates governing equations for u, v, w, π', θ', and q_χ, where $\pi \equiv (p/p_{00})^{R/c_p}$ is a nondimensional pressure, and q_χ ($\chi = v, l, i$) represents the mixing ratios of moisture variables: q_v is water vapor mixing ratio; q_l is the mixing ratio of liquid water; and q_i is the mixing ratio of solid water (ice). Herein, a superscript prime denotes the perturbation from a base-state value. A base-state variable, by definition, is invariant in time and is a function of z only, and is denoted herein by a subscript 0. Thus, a generic variable α may be defined as follows: $\alpha(x, y, z, t) = \alpha_0(z) + \alpha'(x, y, z, t)$. The base state is further assumed to be in hydrostatic balance,

$$\frac{d\pi_0}{dz} = -\frac{g}{c_p \theta_{\rho 0}}, \tag{1}$$

where θ_ρ is density potential temperature,

$$\theta_\rho = \theta \left(\frac{1 + q_v/\varepsilon}{1 + q_v + q_l + q_i} \right). \tag{2}$$

The equation of state is

$$p = \rho R T \left(1 + q_v/\varepsilon\right), \tag{3}$$

or, because $T = \theta\pi$, the equation of state may be equivalently stated as,

$$\pi = \left(\frac{\rho R \theta \left(1 + q_v/\varepsilon\right)}{p_{00}} \right)^{\frac{R}{c_v}}. \tag{4}$$

Note that base state variables must also obey the equation of state. Herein, ρ represents the density of dry air.

The governing equations for velocity are

$$\frac{\partial u}{\partial t} + c_p \theta_\rho \frac{\partial \pi'}{\partial x} = \mathrm{ADV}(u) + fv + T_u + D_u + N_u \tag{5a}$$
$$\frac{\partial v}{\partial t} + c_p \theta_\rho \frac{\partial \pi'}{\partial y} = \mathrm{ADV}(v) - fu + T_v + D_v + N_v \tag{5b}$$
$$\frac{\partial w}{\partial t} + c_p \theta_\rho \frac{\partial \pi'}{\partial z} = \mathrm{ADV}(w) + B + T_w + D_w + N_w \tag{5c}$$

where ADV is the advection operator, formulated in CM1 for a generic variable α as

$$\mathrm{ADV}(\alpha) = \frac{1}{\rho_0}\left[-\frac{\partial(\rho_0 u \alpha)}{\partial x} - \frac{\partial(\rho_0 v \alpha)}{\partial y} - \frac{\partial(\rho_0 w \alpha)}{\partial z} + \alpha\left(\frac{\partial(\rho_0 u)}{\partial x} + \frac{\partial(\rho_0 v)}{\partial y} + \frac{\partial(\rho_0 w)}{\partial z} \right)\right], \tag{6}$$

B is buoyancy,

$$B = g\frac{\theta_\rho - \theta_{\rho 0}}{\theta_{\rho 0}}, \tag{7}$$

and where T terms represent tendencies from subgrid turbulence (see Section 4), the D terms represent optional tendencies from other diffusive processes (discussed at the end of this section), and N represents Newtonian relaxation (i.e., Rayleigh damping). An f-plane is assumed when Coriolis acceleration is included (`icor = 1`).

The governing equations for the three moisture components are

$$\frac{\partial q_v}{\partial t} = \mathrm{ADV}(q_v) + T_{qv} + D_{qv} - \dot{q}_{\mathrm{cond}} - \dot{q}_{\mathrm{dep}}\,, \tag{8a}$$
$$\frac{\partial q_l}{\partial t} = \mathrm{ADV}(q_l) + T_{ql} + D_{ql} + \dot{q}_{\mathrm{cond}} - \dot{q}_{\mathrm{frz}} + \frac{1}{\rho}\frac{\partial(\rho V_l q_l)}{\partial z}\,, \tag{8b}$$
$$\frac{\partial q_i}{\partial t} = \mathrm{ADV}(q_i) + T_{qi} + D_{qi} + \dot{q}_{\mathrm{dep}} + \dot{q}_{\mathrm{frz}} + \frac{1}{\rho}\frac{\partial(\rho V_i q_i)}{\partial z}\,. \tag{8c}$$

The $\dot{q}$ terms represent phase changes between these three components, the T terms represent tendencies from subgrid turbulence (see Section 4), D represents other optional diffusive tendencies (see final paragraph of this section). The last term on the right sides of (8b) and

(8c) represents hydrometeor fallout by a terminal fall velocity (V, which is assumed to be positive-definite).

The governing equation for θ' is

$$\frac{\partial \theta'}{\partial t} = \mathrm{ADV}(\theta) - \Theta_1 \theta \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + T_\theta + D_\theta + N_\theta$$
$$+ \Theta_2 \left(L_v \dot{q}_{\mathrm{cond}} + L_s \dot{q}_{\mathrm{dep}} + L_f \dot{q}_{\mathrm{frz}} \right) + \Theta_3 \left(\dot{q}_{\mathrm{cond}} + \dot{q}_{\mathrm{dep}} \right) + \Theta_2 \epsilon + \dot{Q}_\theta + W_T \quad (9)$$

where T_θ is the tendency from subgrid turbulence (see Section 4), D_θ represents optional diffusive tendencies (discussed at the end of this section), and term $\dot{Q}_\theta$ represents external tendencies to internal energy (primarily radiative heating/cooling). The N_θ term represents the tendency from Newtonian relaxation (i.e., Rayleigh damping), and the W_T term represents the cooling/warming effect from hydrometeors that fall relative to air (i.e., when $V_\chi \neq 0$); most numerical models neglect this effect but it is available in CM1 by setting `efall` $= 1$. The term with ϵ in (9) is associated with dissipative heating, which is the increase in internal energy that occurs when kinetic energy is dissipated; many numerical models neglect this effect but it is available in CM1 by setting `idiss` $= 1$.

The governing equations for π' is

$$\frac{\partial \pi'}{\partial t} - \frac{g}{c_p \theta_{\rho 0}} w + \Pi_1 \pi \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = \mathrm{ADV}(\pi') + \Pi_2 \left(L_v \dot{q}_{\mathrm{cond}} + L_s \dot{q}_{\mathrm{dep}} + L_f \dot{q}_{\mathrm{frz}} \right)$$
$$+ \Pi_3 \left(\dot{q}_{\mathrm{cond}} + \dot{q}_{\mathrm{dep}} \right) + \Pi_4 \left(T_\theta + D_\theta + N_\theta + \Theta_2 \epsilon + \dot{Q}_\theta + W_T \right) + \Pi_5 \left(T_{qv} + D_{qv} \right). \quad (10)$$

In (9) and (10), the variables Θ and Π depend on the value chosen for `neweqts`, and determines whether the equation set mathematically conserves mass and energy. For `neweqts`=0,

$$\Theta_1 = 0, \qquad \Theta_2 = \frac{1}{c_p \pi}, \qquad \Theta_3 = 0, \quad (11)$$

$$\Pi_1 = \frac{R}{c_v}, \qquad \Pi_2 = \Pi_3 = \Pi_4 = \Pi_5 = 0. \quad (12)$$

This option yields the traditional (nonconservative) equation set that is used in many compressible nonhydrostatic models (such as ARPS, MM5, and the Klemp-Wilhelmson Model). The governing equation for θ under this option is equivalent to the one used in the Advanced Research WRF Model (ARW).

For `neweqts` ≥ 1,

$$\Theta_1 = \left(\frac{R_m}{c_{vm}} - \frac{R c_{pm}}{c_p c_{vm}} \right), \qquad \Theta_2 = \frac{c_v}{c_{vm} c_p \pi}, \qquad \Theta_3 = -\theta \frac{R_v}{c_{vm}} \left(1 - \frac{R c_{pm}}{c_p R_m} \right), \tag{13}$$

$$\Pi_1 = \frac{R}{c_p} \frac{c_{pm}}{c_{vm}}, \qquad \Pi_2 = \frac{R}{c_p} \left(\frac{1}{c_{vm} \theta} \right), \qquad \Pi_3 = -\frac{R}{c_p} \left(\pi \frac{R_v c_{pm}}{R_m c_{vm}} \right),$$

$$\Pi_4 = \frac{R}{c_v} \frac{\pi}{\theta}, \qquad \Pi_5 = \frac{R}{c_v} \frac{\pi}{\epsilon + q_v}. \tag{14}$$

where

$$c_{pm} = c_p + c_{pv} q_v + c_l q_l + c_i q_i, \qquad c_{vm} = c_v + c_{vv} q_v + c_l q_l + c_i q_i, \qquad R_m = R + R_v q_v. \tag{15}$$

This option yields the mass- and energy-conserving equations of Bryan and Fritsch (2002). Note that (13) reduces to (11) and (14) reduces to (12) by setting $c_{pv} = c_{vv} = c_l = c_i = R_v = \Pi_2 = \Pi_3 = \Pi_4 = \Pi_5 = 0$.

The latent heats, L, are temperature-dependent according to Kirchoff's relations,

$$\frac{DL_v}{DT} = c_{pv} - c_l, \qquad \frac{DL_s}{DT} = c_{pv} - c_i, \qquad \frac{DL_f}{DT} = c_l - c_i. \tag{16}$$

Numerical values for L are obtained by integration of (16) using reference values $L_v(T_0)$ and $L_v(T_0)$ (see Table 2).

The equations in this section are presented in the exact form that they are integrated in CM1 code. Users should be able to compare directly equations written herein with CM1 code. Note, however, that equations for the axisymmetric version of the model are slightly different; details can be found in Bryan and Rotunno (2009).

Finally, the optional diffusive tendencies represented by D terms above are excluded from most simulations. They can include sixth-order diffusion, which is sometimes used to filter small-scale fluctuations smaller than $\sim$6 times the grid spacing (e.g., Bryan 2005; Knievel et al. 2007). For other types of simulations, this term represents standard viscous stress terms for momentum and conductivity for temperature, which are appropriate for very high resolution Direct Numerical Simulations (DNS) (e.g., Rotunno et al. 2011; Bryan and Rotunno 2014).

3 Terrain

CM1 uses terrain-following coordinates, following Gal-Chen and Somerville (1975). The nominal heights of the coordinate surfaces are given by

$$\sigma = \frac{z_t\left(z - z_s\right)}{z_t - z_s} \tag{17}$$

where $z_s(x, y)$ is the terrain elevation, and z_t is the constant height of the model top. The following metric terms are used in CM1 to account for the coordinate transformation:

$$\begin{aligned} G_x =& \frac{\partial \sigma}{\partial x} = \frac{\sigma - z_t}{z_t - z_s}\frac{\partial z_s}{\partial x} \\ G_y =& \frac{\partial \sigma}{\partial y} = \frac{\sigma - z_t}{z_t - z_s}\frac{\partial z_s}{\partial y} \\ G_z =& \frac{\partial \sigma}{\partial z} = \frac{z_t}{z_t - z_s}. \end{aligned} \tag{18}$$

Horizontal gradients in Cartesian space (e.g., $\partial/\partial x|_z$) can be calculated from gradients along the terrain-following computational coordinates ($\partial/\partial x|_\sigma$) plus "correction terms" as follows:

$$\begin{aligned} \left.\frac{\partial \alpha}{\partial x}\right|_z &= G_z \frac{\partial}{\partial x}\left.\left(\frac{\alpha}{G_z}\right)\right|_\sigma + \frac{\partial}{\partial \sigma}\left(G_x \alpha\right) \\ \left.\frac{\partial \alpha}{\partial y}\right|_z &= G_z \frac{\partial}{\partial y}\left.\left(\frac{\alpha}{G_z}\right)\right|_\sigma + \frac{\partial}{\partial \sigma}\left(G_y \alpha\right). \end{aligned} \tag{19}$$

Vertical gradients are calculated simply by

$$\frac{\partial \alpha}{\partial z} = G_z \frac{\partial \alpha}{\partial \sigma}. \tag{20}$$

For the normal component of velocity to vanish at the surface, the following must hold:

$$w = u\frac{\partial z_s}{\partial x} + v\frac{\partial z_s}{\partial y} \qquad \text{at} \qquad \sigma = 0. \tag{21}$$

From (18) and (21) it follows that

$$\dot{\sigma} \equiv uG_x/G_z + vG_y/G_z + w = 0 \qquad \text{at} \qquad \sigma = 0. \tag{22}$$

Further, it is convenient to formulate the advection operator (6) for simulations with terrain as follows:

$$\begin{aligned} \mathrm{ADV}(\alpha) = \frac{G_z}{\rho_0}\Bigg[- \left.\frac{\partial\left(\alpha\rho_0 u/G_z\right)}{\partial x}\right|_\sigma - \left.\frac{\partial\left(\alpha\rho_0 v/G_z\right)}{\partial y}\right|_\sigma - \frac{\partial\left(\alpha\rho_0\dot{\sigma}\right)}{\partial\sigma} \\ +\alpha\left(\left.\frac{\partial\left(\rho_0 u/G_z\right)}{\partial x}\right|_\sigma + \left.\frac{\partial\left(\rho_0 v/G_z\right)}{\partial y}\right|_\sigma + \frac{\partial\left(\rho_0\dot{\sigma}\right)}{\partial\sigma}\right)\Bigg]. \end{aligned} \tag{23}$$

4 Subgrid turbulence

If a subgrid turbulence scheme is used (`iturb` $\geq$ `1`) then the tendencies from subgrid turbulence (represented generically by the T terms in Section 2) are formulated as follows:

$$T_u = \frac{1}{\rho}\left[\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} + \frac{\partial \tau_{13}}{\partial z}\right] \tag{24}$$

$$T_v = \frac{1}{\rho}\left[\frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{23}}{\partial z}\right] \tag{25}$$

$$T_w = \frac{1}{\rho}\left[\frac{\partial \tau_{13}}{\partial x} + \frac{\partial \tau_{23}}{\partial y} + \frac{\partial \tau_{33}}{\partial z}\right] \tag{26}$$

$$T_s = -\frac{1}{\rho}\left[\frac{\partial \tau_1^s}{\partial x} + \frac{\partial \tau_2^s}{\partial y} + \frac{\partial \tau_3^s}{\partial z}\right] \tag{27}$$

where s represents one of the model scalars (θ, q_v, q_l, or q_i). The subgrid stress terms (τ_{ij}) are formulated as follows:

$$\tau_{ij} \equiv \overline{\rho u_i' u_j'} = 2\rho K_m S_{ij}, \tag{28}$$

where K_m is an eddy viscosity (see below) and S_{ij} is the mean strain tensor,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{29}$$

The turbulent fluxes for scalars are parameterized as:

$$\tau_i^{\theta} \equiv \overline{\rho u_i' \theta'} = -\, K_h \rho \frac{\partial \theta}{\partial x_i}, \tag{30}$$

$$\tau_i^{q_v} \equiv \overline{\rho u_i' q_v'} = -\, K_h \rho \frac{\partial q_v}{\partial x_i}, \tag{31}$$

$$\tau_i^{q_l} \equiv \overline{\rho u_i' q_l'} = -\, K_h \rho \frac{\partial q_l}{\partial x_i}, \tag{32}$$

$$\tau_i^{q_i} \equiv \overline{\rho u_i' q_i'} = -\, K_h \rho \frac{\partial q_i}{\partial x_i}, \tag{33}$$

where K_h is an eddy diffusivity (see below). The relations (28) and (30)–(33) apply to the interior of a model domain; different formulations are applied on boundaries to account for surface stress (i.e., drag) and fluxes of temperature and moisture.

The method to determine K_m and K_h depends on the type of closure chosen. For `iturb = 1` the subgrid turbulence kinetic energy (TKE) is predicted and used to determine K_m and K_h. The scheme in CM1 is similar to that described by Deardorff (1980). The eddy viscosity K_m is determined from the relation

$$K_m = c_m l e^{1/2}. \tag{34}$$

and the eddy diffusivity K_h is determined from the relation

$$K_h = c_h l e^{1/2}, \tag{35}$$

where $e = \frac{1}{2}\overline{u_i' u_i'}$ is the subgrid TKE. The predictive equation for e is

$$\frac{\partial e}{\partial t} = \text{ADV}(e) + K_m S^2 - K_h N_m^2 + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(2\rho K_m \frac{\partial e}{\partial x_i}\right) - \epsilon \tag{36}$$

where ϵ is dissipation, which is parameterized as

$$\epsilon = c_\epsilon e^{3/2}/l, \tag{37}$$

and S^2 is deformation,

$$S^2 = 2 S_{ij} S_{ij}. \tag{38}$$

N_m^2 is the squared Brunt-Väisälä frequency, which for subsaturated air is given by

$$N_m^2 = \frac{g}{\theta_\rho}\frac{\partial \theta_\rho}{\partial z}, \tag{39}$$

and for saturated air is given by

$$N_m^2 = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \Gamma_m\right)\left(1 + \frac{T}{\varepsilon + q_s}\frac{\partial q_s}{\partial T}\right) - \frac{g}{1 + q_t}\frac{\partial q_t}{\partial z}, \tag{40}$$

where q_s is saturation mixing ratio, Γ_m is the moist-adiabatic lapse rate,

$$\Gamma_m = g(1 + q_t)\left(\frac{1 + L_v q_s / RT}{c_{pm} + L_v \partial q_s / \partial T}\right). \tag{41}$$

and $q_t = 1 + q_v + q_l + q_i$ is the total water mixing ratio.

The parameters c_m, c_h, c_ϵ, and l must be specified to close these equations. For details, see Appendix B in Stevens et al. (1999). In CM1, the default value for c_m is 0.10. The parameters c_h, c_ϵ, and l have a stability dependence that is designed to reduce subgrid-scale mixing in statically stable conditions (i.e., for $N_m^2 > 0$). The default formulation in CM1 is as follows:

$$
\begin{aligned}
c_h =& 1 + 2\frac{l}{\Delta} && (42)\\
c_\epsilon =& 0.2 + 0.787\frac{l}{\Delta} && (43)\\
l =& \left(\frac{2}{3}\frac{e}{N_m^2}\right)^{1/2}, && (44)
\end{aligned}
$$

where Δ is a measure of the grid size, e.g.,

$$\Delta = (\Delta x \, \Delta y \, \Delta z)^{1/3} \,. \tag{45}$$

Note that $l = \Delta$ is used for $N_m^2 \leq 0$. The settings for c_m, c_ϵ, and l in CM1 ensure that turbulence is inactive (i.e., $K_m = K_h = 0$) when $\mathrm{Ri} > 0.25$, where Ri is the Richardson number,

$$\mathrm{Ri} = \frac{N_m^2}{S^2}. \tag{46}$$

For `iturb = 2` a simpler scheme is used. By assuming steady and homogeneous subgrid turbulence, and by neglecting the stability dependence of the parameters discussed in the previous paragraph, then the following relation can be derived:

$$K_m = (C_s \Delta)^2 \left[S^2 \left(1 - \frac{\mathrm{Ri}}{\mathrm{Pr}} \right) \right]^{1/2}, \tag{47}$$

where $C_s = 0.18$ is the Smagorinsky constant [after Smagorinsky (1963)] and $\mathrm{Pr} \approx 1/3$ is the Prandtl number. The eddy diffusivity is given simply by

$$K_h = K_m/\mathrm{Pr}. \tag{48}$$

If $\mathrm{Ri} > \mathrm{Pr}$ in (47) then K_m is set to zero; hence, subgrid turbulence is active (i.e., $K_m > 0$) only when $\mathrm{Ri} < \mathrm{Pr}$.

Compared to the TKE scheme, the Smagorinsky scheme has three primary disadvantages: 1) Subgrid turbulence for the Smagorinsky scheme is active (i.e., $K_m > 0$) only when $S^2 > 0$ (i.e., in locally sheared conditions). 2) The assumption of steady and isotropic turbulence inherent in the Smagorinsky scheme is a major disadvantage in some situations, particularly when resolution is poor. 3) As formulated, there is no stability dependence to the inherent length scales in this Smagorinsky scheme, which makes it too diffusive in stable conditions ($N_m^2 > 0$). This last deficiency can be alleviated (see, e.g., Stevens et al. 1999), and might be addressed in some future version of CM1.

When using `iturb` = 1 or `iturb` = 2, it is assumed that some turbulent eddies are resolved explicitly during the simulation; i.e., these schemes are appropriate for large eddy simulation (LES). For `iturb` = 3, it is assumed that no turbulent eddies are resolved on the grid, and hence their affects must be accounted completely via the T terms in section 2. CM1 uses a scheme that is similar to the Smagorinsky scheme except different eddy viscosities must be used for the horizontal and vertical directions, and they are formulated as follows:

$$K_{m,h} = l_h^2 S_h \qquad \text{and} \qquad K_{m,v} = l_v^2 S_v \left(1 - \frac{\mathrm{Ri}}{\mathrm{Pr}}\right)^{1/2} \tag{49}$$

where l_h and l_v are lengthscales for subgrid turbulence in the horizontal and vertical directions, respectively, and where

$$S_h{}^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2, \tag{50}$$

$$S_v{}^2 = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2. \tag{51}$$

Starting with `cm1r17`, l_v is determined from the relation $l_v^{-2} = (\kappa z)^{-2} + l_\infty^{-2}$ where l_∞ is a specified asympototic length scale far from the surface. The values for l_h and l_∞ must be chosen carefully; they scale roughly with the largest turbulent eddies in a flow. The value for the Prandtl number Pr must also be set carefully, but in CM1 it is set to 1 by default.

For the axisymmetric version of CM1, only `iturb = 3` can be used. Details are available in Bryan and Rotunno (2009).

5 Anelastic/incompressible equations

The equations in section 2 are used in CM1 when one of the compressible solvers are chosen (`psolver` = 1,2,3). CM1 also has the ability to use the anelastic equations (`psolver` = 4) and the incompressible equations (`psolver` = 5).

For the anelastic equations, the velocity equations can be written as:

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = F_u \tag{52a}$$
$$\frac{\partial v}{\partial t} + \frac{\partial \phi}{\partial y} = F_v \tag{52b}$$
$$\frac{\partial w}{\partial t} + \frac{\partial \phi}{\partial z} = F_w. \tag{52c}$$

where F_u, F_v, and F_w represent all terms on the right side of (55), (5b), and (57), respectively. Notice that the pressure-gradient terms are written in terms of $\phi \equiv p'/\rho_0$. There is no predictive equation for pressure in this system of equations. Hence, (10) is not integrated in the anelastic system. Instead, a diagnostic equation for ϕ is developed by using the anelastic mass-continuity equation,

$$\frac{\partial}{\partial x_i}\left(\rho_0 u_i\right) = 0. \tag{53}$$

Using (52) and (53), the diagnostic equation for ϕ is simply

$$\frac{\partial}{\partial x_i}\left(\rho_0 \frac{\partial \phi}{\partial x_i}\right) = \frac{\partial\left(\rho_0 F_u\right)}{\partial x} + \frac{\partial\left(\rho_0 F_v\right)}{\partial y} + \frac{\partial\left(\rho_0 F_w\right)}{\partial z}. \tag{54}$$

CM1 solves (54) using a direct method based on fast Fourier transforms. Because the anelastic equations do not permit acoustic waves, there is no need for small time steps.

The incompressible equations are the same as the anelastic equations except it is assumed that ρ_0 = constant. This system of equations is only appropriate for simulations with a shallow domain (of order 1 km or less).

6 Compressible-Boussinesq equations

In cm1r18, there is an option to use the "compressible Boussinesq" equations (`psolver = 6`). In this equation set, the Boussinesq approximation has been made, in which density variations are neglected in the velocity equations (except where multiplied by gravity). Further, a prognostic "pressure" equation is used, and thus acoustic waves are permitted. The equations are as follows:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{\partial \phi}{\partial x} \tag{55}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{\partial \phi}{\partial y} \tag{56}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{\partial \phi}{\partial z} + g\frac{\theta'}{\theta_0} \tag{57}$$

$$\frac{\partial \phi}{\partial t} = -c_s^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) . \tag{58}$$

In the final equation, c_s is a constant speed of sound, specified by the user. The default value is 300 m s^{-1}. These equations are most applicable to shallow flows (of order 1 km) and should only be used in similar situations as the incompressible equations. These equations are solved using the Klemp-Wilhelmson time-splitting scheme with explicit calculations in both horizontal and vertical directions (the same as `psolver = 2`).

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Table 1: Variables and arrays in CM1.

Symbol	Description	Name in code
Predicted variables:		
q	Mixing ratio of moisture	`qa` (at t)
		`q3d` (at $t+\Delta t$)
u	Velocity in x	`ua` (at t)
		`u3d` (at $t+\Delta t$)
v	Velocity in y	`va` (at t)
		`v3d` (at $t+\Delta t$)
w	Velocity in z	`wa` (at t)
		`w3d` (at $t+\Delta t$)
θ_0	Base-state θ	`th0`
θ'	Perturbation θ	`tha` (at t)
		`th3d` (at $t+\Delta t$)
π_0	Base-state π	`pi0`
π'	Perturbation π	`ppi` (at t)
		`pp3d` (at $t+\Delta t$)
Derived variables:		
p	Pressure	`prs`
T	Temperature ($T=\theta\pi$)	varies
θ	Potential temperature ($\theta=\theta_0+\theta'$)	varies
π	Nondimensional pressure ($\pi=\pi_0+\pi'$)	varies
ρ	Density of dry air	`rho`
$\rho_0 u$	u multiplied by ρ_0	`rru` (if no terrain)
$\rho_0 v$	v multiplied by ρ_0	`rrv` (if no terrain)
$\rho_0 w$	w multiplied by ρ_0	`rrw` (if no terrain)
$\rho_0 u/G_z$	$\rho_0 u/G_z$	`rru` (if terrain)
$\rho_0 v/G_z$	$\rho_0 v/G_z$	`rrv` (if terrain)
$\rho_0\dot{\sigma}$	$\rho_0\dot{\sigma}=\rho_0\left(uG_x/G_z+vG_y/G_z+w\right)$	`rrw` (if terrain)
Variables for terrain only:		
G_x	$\frac{\sigma-z_t}{z_t-z_s}\frac{\partial z_s}{\partial x}$	`gx`
G_y	$\frac{\sigma-z_t}{z_t-z_s}\frac{\partial z_s}{\partial y}$	`gy`
G_z	$\frac{z_t}{z_t-z_s}$	`gz`
z_s	Terrain height	`zs`
z_t	Height at top of domain	`zt`
σ	Nominal height of model levels, $\sigma=\frac{z_t(z-z_s)}{z_t-z_s}$	`sigma`

Table 2: Constants in CM1. See `constants.incl` for values.

Symbol	Description	Name in code
c_i	Specific heat of ice	`cpi`
c_l	Specific heat of liquid water	`cpl`
c_p	Specific heat of dry air at constant pressure	`cp`
c_{pv}	Specific heat of water vapor at constant pressure	`cpv`
c_v	Specific heat of dry air at constant volume	`cv`
c_{vv}	Specific heat of water vapor at constant volume	`cvv`
f	Coriolis parameter	`fcor`
g	Gravitational acceleration	`g`
$L_v(T_0)$	Reference value of L_v at $T = T_0$	`xlv`
$L_s(T_0)$	Reference value of L_s at $T = T_0$	`xls`
p_{00}	Reference pressure	`p00`
R	Gas constant for dry air	`rd`
R_v	Gas constant for water vapor	`rv`
T_0	Reference temperature	`to`
ε	Ratio of gas constants: R/R_v	`eps`
κ	von Karman constant	`karman`