Scalar advection with adaptive moving meshes

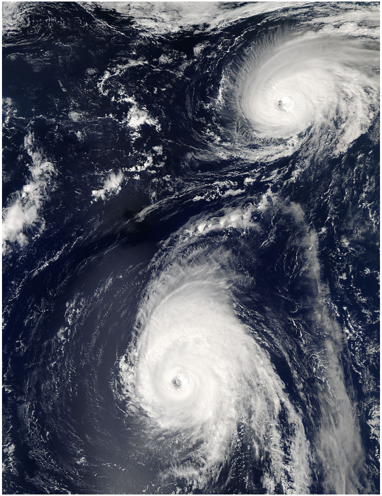
Christian Kühnlein and Andreas Dörnbrack

DLR, Institut für Physik der Atmosphäre

Bad Tölz | October 7, 2008

Motivation

- Atmospheric flows: Interaction of a large range of different scales.
- Adequate resolution of the various scales important for the dynamics requires extreme computational effort.
- Use of dynamical adaptive resolution technique could provide one possibility to reduce computational cost in order to achieve a certain resolution of the overall flow.



Project

- DFG-MetStröm priority programme.
- Apply solution-adaptive resolution technique in EULAG for simulating atmospheric flows.
- Extend recent work of Smolarkiewicz and Prusa (2003) by using *a posteriori* criteria and mesh differential equations for adaption in EULAG.

▶ Here, presentation of the methodology by means of application to 2D linear advection problem with solver MPDATA.

Outline

1. Introduction
2. Formulation of the linear advection model based on design of EULAG
3. Dynamic grid adaption
4. Example application: Rotating cone

Mesh adaption approach: Continuous mappings

Continuous mapping approach: Adaptive curvilinear grid in physical domain S_p specified by time-variable mapping

$$\mathbf{x}(\overline{\mathbf{x}}, \overline{t}) : S_t \rightarrow S_p$$

where S_t transformed domain with reference computational grid.

- Mapping may be specified either analytically or numerically.
- Redistribution, no insertion/deletion of grid points $\Rightarrow$ conserved data structures.
- Physical problem is formulated in transformed domain S_t.

DLR

Prototype physical test problem

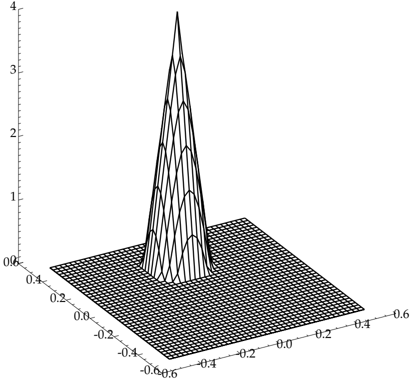
Linear scalar advection in two dimensions

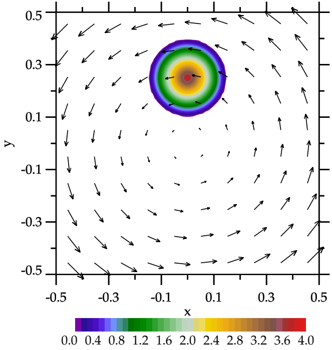
$$\frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} = 0$$

- $\psi(x, y, t)$: Passive scalar function of Cartesian coordinates x, y
- $\mathbf{v} = (u, v)^T$: Spatially-variable solenoidal velocity field
- Exact solution $\psi(x, y, t) = \psi(x - ut, y - vt, 0)$.

Prototype physical test problem

Benchmark: Rotating cone





Model formulation: Coordinate mapping

General time-dependent coordinate mapping of 3D solver EULAG:

$$(\overline{x}, \overline{y}, \overline{z}, \overline{t}) = (\overline{x}(x, y, t), \overline{y}(x, y, t), \overline{z}(x, y, z, t), t) : \quad \mathbf{S}_p \rightarrow \mathbf{S}_t$$

Model formulation: Coordinate mapping

General time-dependent coordinate mapping of 3D solver EULAG:

$$(\overline{x}, \overline{y}, \overline{z}, \overline{t}) = (\overline{x}(x, y, t), \overline{y}(x, y, t), \overline{z}(x, y, z, t), t) : \quad \mathbf{S}_p \rightarrow \mathbf{S}_t$$

General coordinate mapping in the 2D model:

$$(\overline{x}, \overline{y}, \overline{t}) = (\overline{x}(x, y, t), \overline{y}(x, y, t), t) : \quad \mathbf{S}_p \rightarrow \mathbf{S}_t$$

Model formulation: Generalised conservation form

Generalised Eulerian conservation form:

$$\frac{\partial(\overline{S}\psi)}{\partial\overline{t}} + \frac{\partial(\overline{S}\overline{u}\psi)}{\partial\overline{x}} + \frac{\partial(\overline{S}\overline{v}\psi)}{\partial\overline{y}} = 0$$

- $\overline{u}, \overline{v}$: contravariant velocity components given as

$$\overline{u} = \frac{d\overline{x}}{d\overline{t}} = \frac{\partial\overline{x}}{\partial t} + u\frac{\partial\overline{x}}{\partial x} + v\frac{\partial\overline{x}}{\partial y} \qquad \overline{v} = \frac{d\overline{y}}{d\overline{t}} = \frac{\partial\overline{y}}{\partial t} + u\frac{\partial\overline{y}}{\partial x} + v\frac{\partial\overline{y}}{\partial y}$$

- $\overline{S}$: Jacobian of the transformation given as

$$\overline{S} = \left(\frac{\partial\overline{x}}{\partial x}\frac{\partial\overline{y}}{\partial y} - \frac{\partial\overline{x}}{\partial y}\frac{\partial\overline{y}}{\partial x}\right)^{-1}$$

Model formulation: Tensor identities

$$\frac{\partial}{\partial \overline{x}}\left(\overline{S}\frac{\partial \overline{x}}{\partial x}\right)+\frac{\partial}{\partial \overline{y}}\left(\overline{S}\frac{\partial \overline{y}}{\partial x}\right)=0$$

$$\frac{\partial}{\partial \overline{x}}\left(\overline{S}\frac{\partial \overline{x}}{\partial y}\right)+\frac{\partial}{\partial \overline{y}}\left(\overline{S}\frac{\partial \overline{y}}{\partial y}\right)=0$$

$$\frac{\partial \overline{S}}{\partial \overline{t}}+\frac{\partial}{\partial \overline{x}}\left(\overline{S}\frac{\partial \overline{x}}{\partial t}\right)+\frac{\partial}{\partial \overline{y}}\left(\overline{S}\frac{\partial \overline{y}}{\partial t}\right)=0.$$

Geometric Conservation Law (GCL)

Prusa et al. (2006)

Model formulation: Computation of the Jacobian

1. Diagnostic Jacobian:

$$\overline{S}_d := \left(\frac{\partial \overline{x}}{\partial x} \frac{\partial \overline{y}}{\partial y} - \frac{\partial \overline{x}}{\partial y} \frac{\partial \overline{y}}{\partial x} \right)^{-1}$$

2. Time-component of the GCL:

$$\frac{\partial \overline{S}}{\partial \overline{t}} + \frac{\partial}{\partial \overline{x}} \left(\overline{S} \frac{\partial \overline{x}}{\partial t} \right) + \frac{\partial}{\partial \overline{y}} \left(\overline{S} \frac{\partial \overline{y}}{\partial t} \right) = 0.$$

Model formulation: Numerical solution

MPDATA for advection in time-dependent geometries:

$$\psi^{n+1} = \frac{\overline{S}^n}{\overline{S}^{n+1}} \left(\psi^n - \frac{\delta t}{\overline{S}^n} \overline{\nabla} \cdot \left(\hat{\mathbf{v}}^{n+1/2} \psi^n \right) \right)$$

$$\psi_{\mathbf{i}}^{n+1} = \frac{\overline{S_{\mathbf{i}}}^n}{\overline{S_{\mathbf{i}}}^{n+1}} \, \mathrm{MPDATA}_{\mathbf{i}} \left(\psi^n, \hat{\mathbf{v}}^{n+1/2}, \overline{S}^n \right)$$

Smolarkiewicz (1999,2006)

Mesh adaption

- Continuously redistribute fixed number grid points according to the evolution of the flow.

- Construct indicators for local amount of adaptivity.
- Distribute available grid points according to these indicators.

Basic principle of mesh adaption: 1D-perspective

- With mapping: $x(\overline{x}) : S_t \longrightarrow S_p$ we estimate

$$x(\overline{x}_{i+1}) - x(\overline{x}_i) = \frac{\partial x}{\partial \overline{x}}\,\delta\overline{x} + \mathcal{O}(\delta\overline{x}^2)$$

- Relationship:

$$\text{grid point density} \sim (x_{i+1} - x_i)^{-1} \sim \left(\frac{\partial x}{\partial \overline{x}}\right)^{-1} = \frac{\partial \overline{x}}{\partial x}$$

- Introduce monitor function $m(x) : S_p \longrightarrow \mathbb{R}^+$ chosen that

$$m(x) \sim \frac{\partial \overline{x}}{\partial x}$$

Basic principle of mesh adaption: 1D-perspective

- ansatz:

$$c \cdot m(x) = \frac{\partial \overline{x}}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial x}\left(m^{-1}(x)\frac{\partial \overline{x}}{\partial x}\right) = 0 \; + BCs$$

- idea: $m(x) \rightarrow \overline{x}(x) \rightarrow x(\overline{x})$

▶ $x(\overline{x})$ satisfies equidistribution condition with respect to the monitor function $m(x)$.

Basic principle of mesh adaption: 1D-perspective

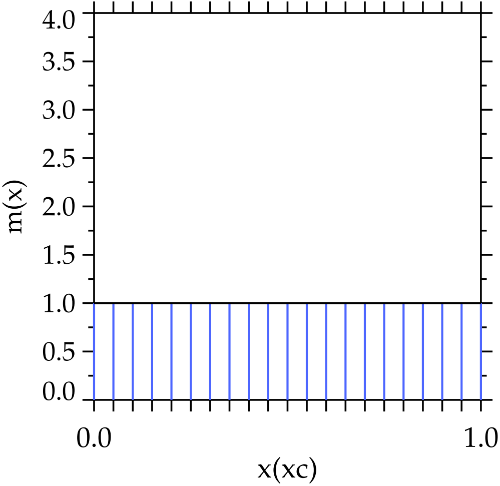
- Example:

  $S_p = [0, 1]$

  $\overline{x}(0) = 0, \ \overline{x}(1) = 1$

  $m(x) = 1$

Number of grid increments: $N = 20$



Basic principle of mesh adaption: 1D-perspective

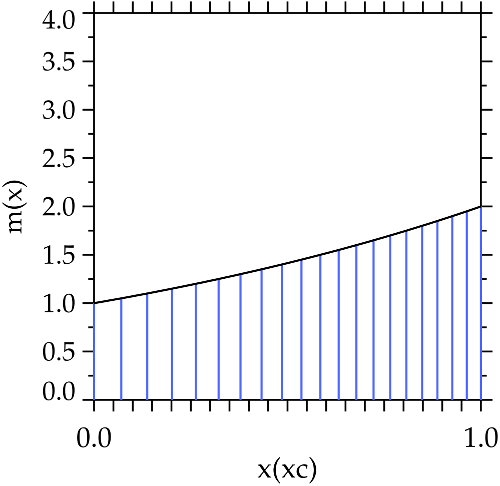
- Example:

  $S_p = [0, 1]$

  $\overline{x}(0) = 0, \;\; \overline{x}(1) = 1$

  $m(x) = \exp\left(x \ln 2\right)$

Number of grid increments: N =20



Basic principle of mesh adaption: 1D-perspective

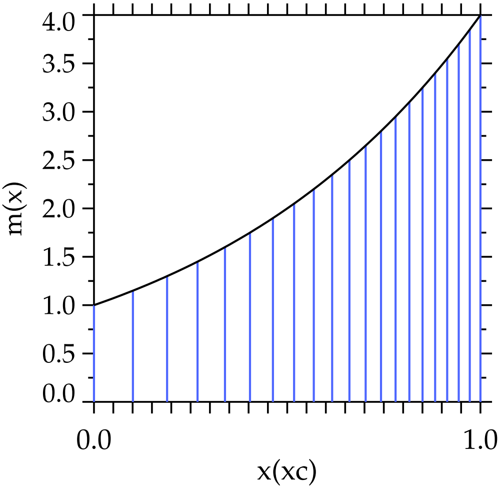
- Example:

  $S_p = [0, 1]$

  $\overline{x}(0) = 0, \ \overline{x}(1) = 1$

  $m(x) = \exp\left(x \ln 4\right)$

Number of grid increments: $N =$20



Basic principle of mesh adaption: 1D-perspective

- Example:

  $S_p = [0, 1]$

  $\overline{x}(0) = 0, \;\; \overline{x}(1) = 1$

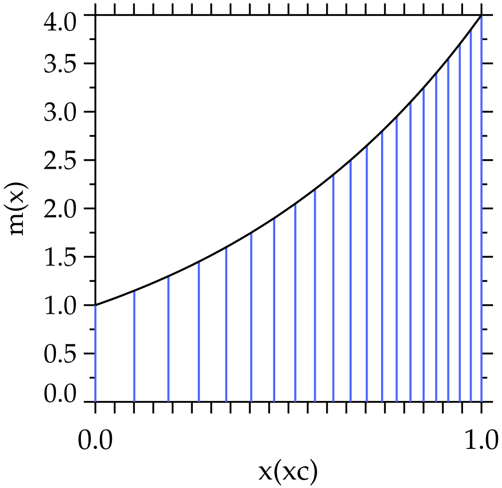
  $m(x) = \exp(x \ln 4)$

$$\int_{x_{i-1}}^{x_i} m(x)dx = c$$

$$i = 1, \ldots, N$$

$$c = \frac{1}{N} \int_0^1 m(x)dx$$

Number of grid increments: $N = 20$



Adaption in two-dimensional space: Variational formulation

Mesh adaption functional:

$$I[\overline{\mathbf{x}}] = \frac{1}{2} \int_{\mathbf{S}_p} \sum_{i=1}^{2} (\nabla \overline{x}^i)^T M^{-1} \nabla \overline{x}^i \, d\mathbf{x}$$

- M denotes 2×2 symmetric positive definite monitor matrix
- M constitutes link to physical solution
- M can also be used to influence the geometric properties of the generated mesh

Adaption in two-dimensional space: Variational formulation

Mesh adaption functional:

$$I[\overline{\mathbf{x}}] = \frac{1}{2} \int_{\mathbf{S}_p} \sum_{i=1}^{2} (\nabla \overline{x}^i)^T \, M^{-1} \, \nabla \overline{x}^i \, d\mathbf{x}$$

- M denotes 2×2 symmetric positive definite monitor matrix
- M constitutes link to physical solution
- M can also be used to influence the geometric properties of the generated mesh

Euler-Lagrange equations:

$$\nabla \cdot (M^{-1} \nabla \overline{x}^i) = 0 \qquad i = 1, 2$$

DLR

Adaption in two-dimensional space: Variational formulation

Mesh adaption functional:

$$I[\overline{\mathbf{x}}] = \frac{1}{2} \int_{\mathbf{S}_p} \sum_{i=1}^{2} (\nabla \overline{x}^i)^T \, M^{-1} \, \nabla \overline{x}^i \, d\mathbf{x}$$

- M denotes 2×2 symmetric positive definite monitor matrix
- M constitutes link to physical solution
- M can also be used to influence the geometric properties of the generated mesh

In one dimension:

$$\frac{\partial}{\partial x} \left(m^{-1} \frac{\partial \overline{x}}{\partial x} \right) = 0$$

Monitor matrix

We define

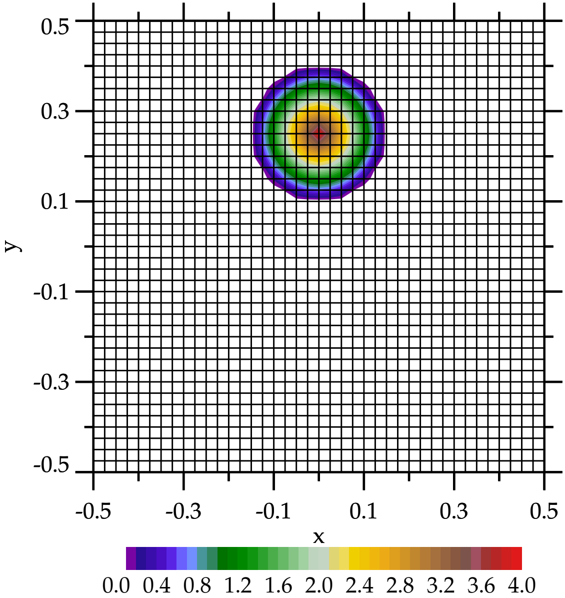
$$M = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

- $q = q(\mathbf{x}, t)$ strictly positive scalar weight function.
- Grid point concentration where q relatively large.
- Local gradient criteria:

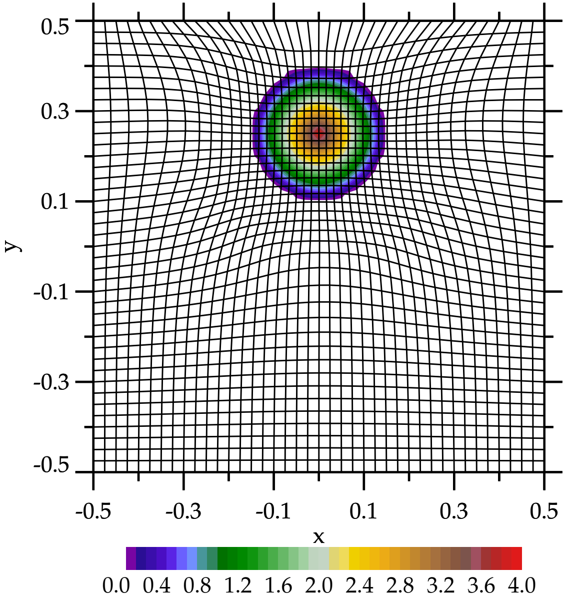
$$q(\mathbf{x}, t) = \alpha + \frac{\beta}{(1-\beta)} \left|\nabla\psi(\mathbf{x}, t)\right|^{\frac{1}{m}} \qquad \beta \in (0.0, 0.95)$$

- Low-pass filtering is applied to q.

Static equidistant mesh

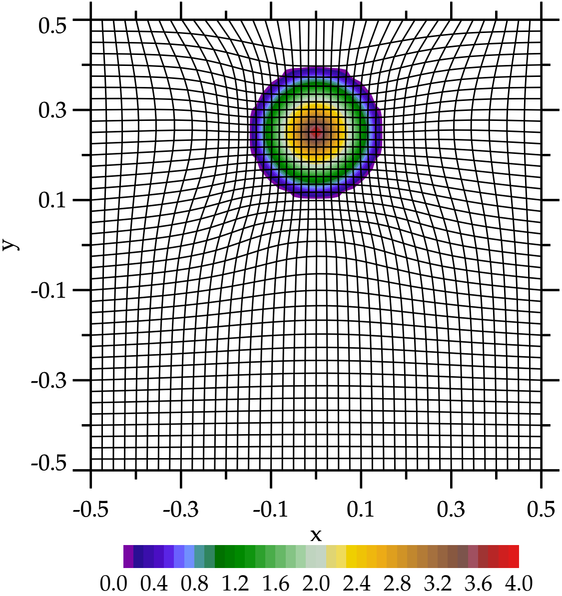


Dynamic adaption

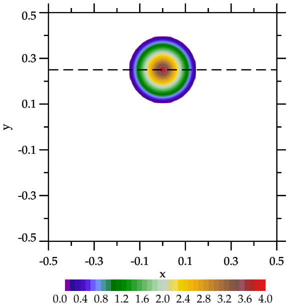


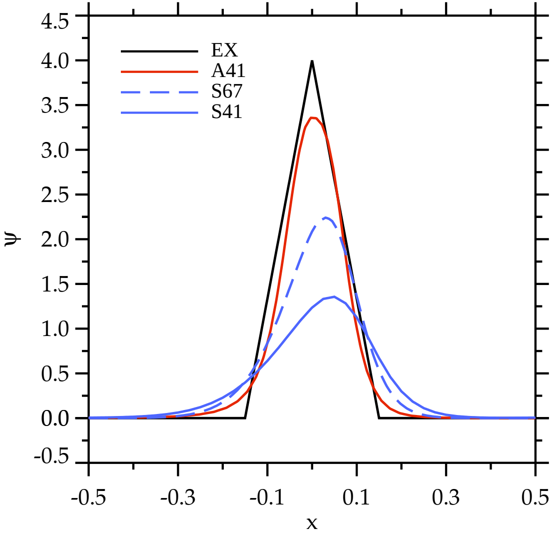
DLR

Dynamic adaption

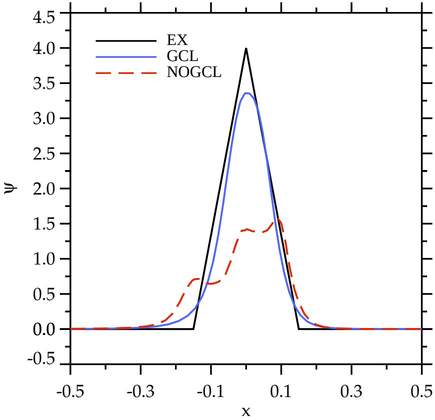


Comparison dynamic adaptive vs. static equidistant mesh

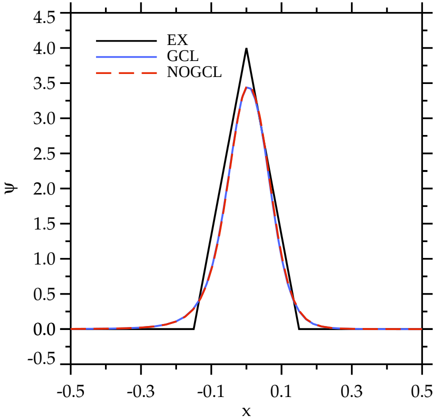




Importance of the geometric conservation law



6 filter passes of monitor function



20 filter passes of monitor function