Gravity Waves, Scale Asymptotics, and the Pseudo-Incompressible Equations

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13.9.10

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Gravity waves
in the
middle atmosphere

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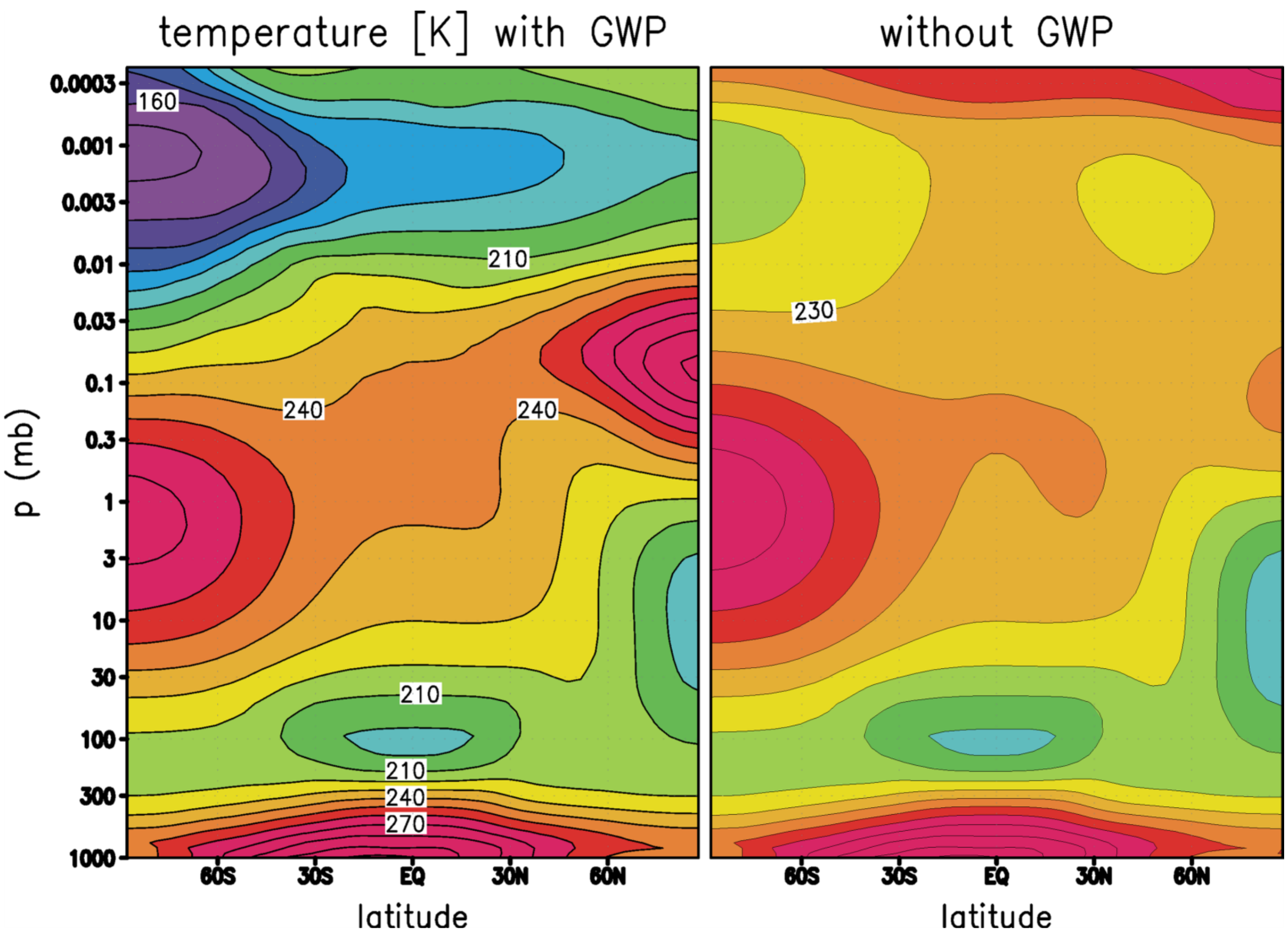
Gravity Waves in the Middle Atmosphere



Becker und Schmitz (2003)

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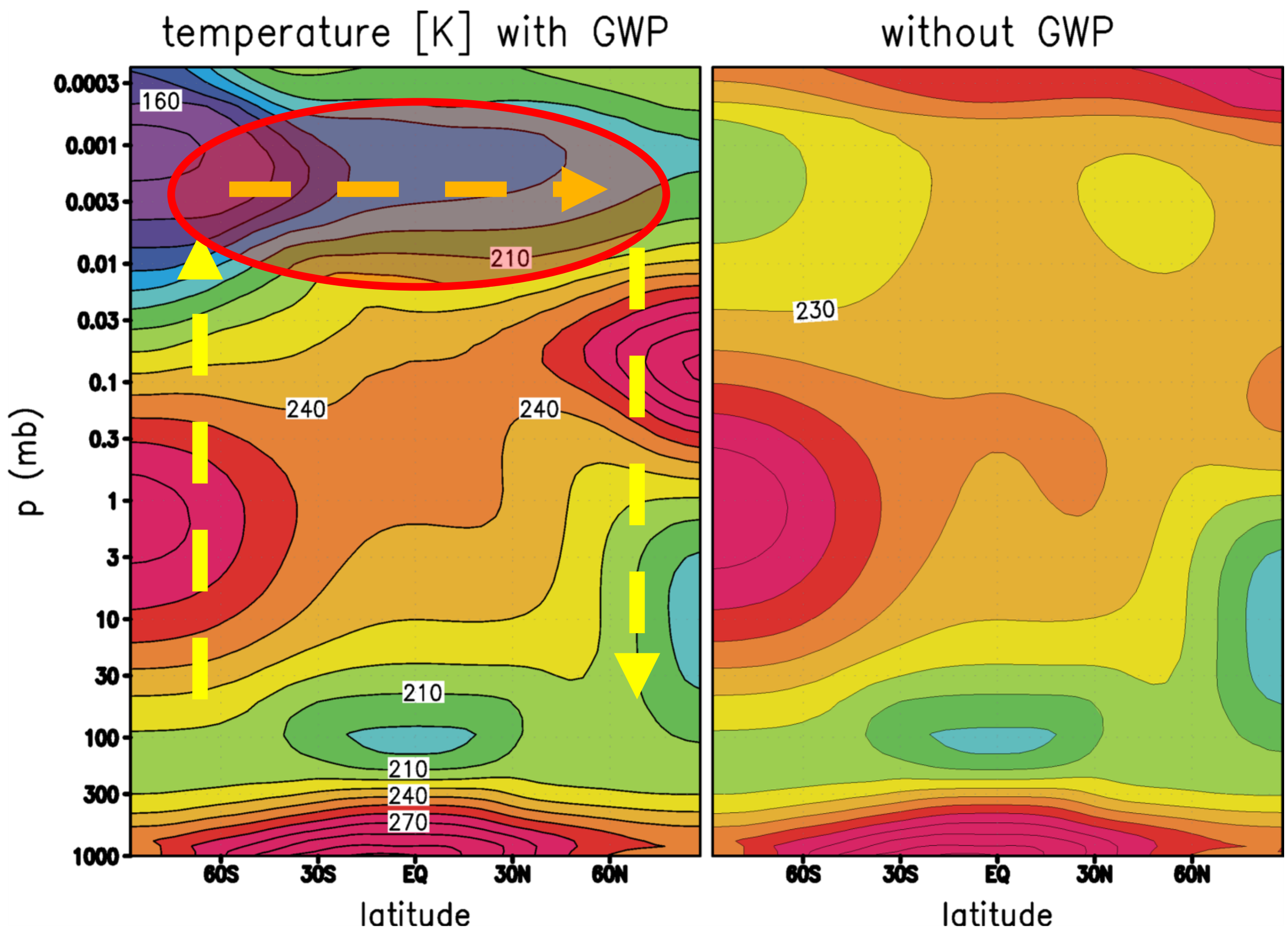

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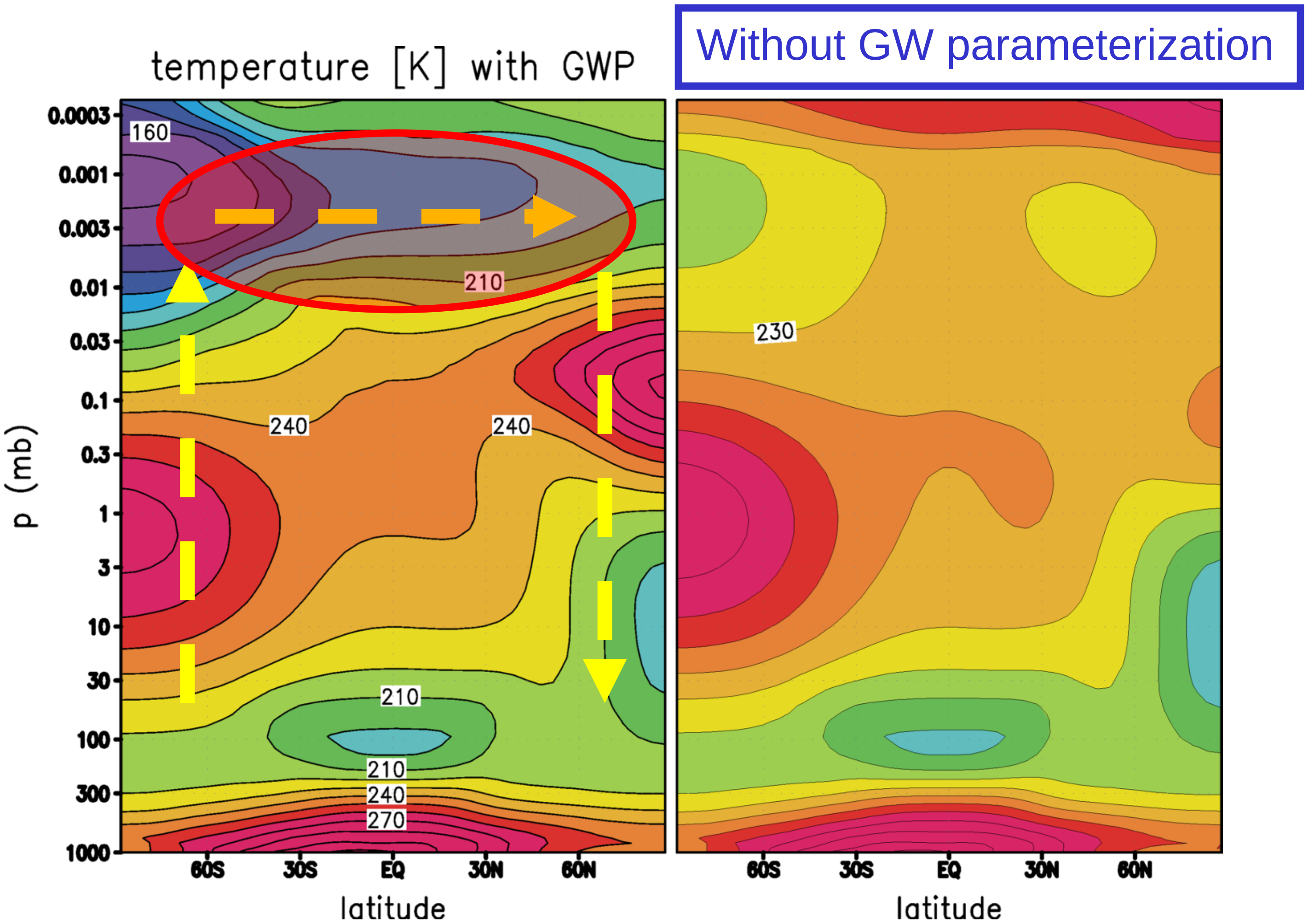

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linear GWs in the Euler equations

- no heat sources, friction, …
- no rotation

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p - g\mathbf{e}_z$$

$$\frac{D\theta}{Dt} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

$$p = \rho RT$$

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

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are equivalent to

$$\frac{D\mathbf{v}}{Dt} = -c_p\theta\nabla\pi - g\mathbf{e}_z$$
$$\frac{D\theta}{Dt} = 0$$
$$\frac{D\pi}{Dt} + \frac{R}{c_v}\pi\nabla\cdot\mathbf{v} = 0$$

$$p = \rho RT$$

$$\pi = \left(\frac{p}{p_0}\right)^{R/c_p}, \qquad \theta = \frac{T}{\pi}$$

linear GWs in the Euler equations

- linearization about atmosphere at rest:

$$\mathbf{v} = \mathbf{v}'$$

$$\theta = \overline{\theta}(z) + \theta'$$

$$\pi = \overline{\pi}(z) + \pi'$$

$$0 = -c_p \overline{\theta} \frac{d\overline{\pi}}{dz} - g \qquad \Rightarrow$$

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$$\frac{D\mathbf{v}'}{Dt} = -c_p \overline{\theta} \nabla \pi' + b' \mathbf{e}_z$$

$$c_p \overline{\theta} \frac{\partial \pi'}{\partial t} - g w' + \frac{R}{c_v} c_p \overline{\theta}\overline{\pi} \nabla \cdot \mathbf{v}' = 0$$

$$\frac{\partial b'}{\partial t} + N^2 w' = 0$$

$$\mathbf{v}' = (u', v', w')$$

$$b' = g \frac{\theta'}{\overline{\theta}}$$

$$N^2 = \frac{g}{\overline{\theta}} \frac{d\overline{\theta}}{dz}$$

linear GWs in the Euler equations

- conservation of wave energy: linear equations satisfy

$$\frac{\partial E'}{\partial t} + \nabla \cdot \left(p' \mathbf{v}' \right) = 0$$

$$E' = \frac{\overline{\rho}}{2} \left(\left| \mathbf{v}' \right|^2 + \frac{b'^2}{N^2} \right) + \frac{\overline{\rho}\overline{\theta}^2}{2c_s^2} c_p^2 \pi'^2 , \qquad c_s^2 = \frac{c_p}{c_v} R\overline{T}$$

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- conservation suggests:

$$\mathbf{v}' \propto 1 / \sqrt{\overline{\rho}}$$

$$b' \propto N / \sqrt{\overline{\rho}}$$

$$\pi' \propto c_s / \left(\overline{\theta} \sqrt{\overline{\rho}} \right)$$

linear GWs in the Euler equations

- isothermal atmosphere: $c_s = \text{const.}, \qquad N^2 = \text{const.}$

$$\mathbf{v}' = \sqrt{\frac{\overline{\rho}_0}{\overline{\rho}}}\,\tilde{\mathbf{v}} \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right] \qquad \mathbf{k} = (k, l, m)$$

$$b' = \sqrt{\frac{\overline{\rho}_0}{\overline{\rho}}}\,\tilde{b} \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]$$

$$\pi' = \sqrt{\frac{\overline{\rho}_0}{\overline{\rho}}}\,\frac{\theta_0}{\overline{\theta}}\,\tilde{\pi} \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]$$

linear GWs in the Euler equations

- isothermal atmosphere: $c_s = \text{const.}, \qquad N^2 = \text{const.}$

$$\omega^2 = \begin{cases} \dfrac{N^2\left(k^2+l^2\right)}{k^2+l^2+m^2+\dfrac{1}{4H^2}} & \text{gravity waves} \\ c_s^2\left(k^2+l^2+m^2+\dfrac{1}{4H^2}\right) & \text{sound waves} \end{cases}$$

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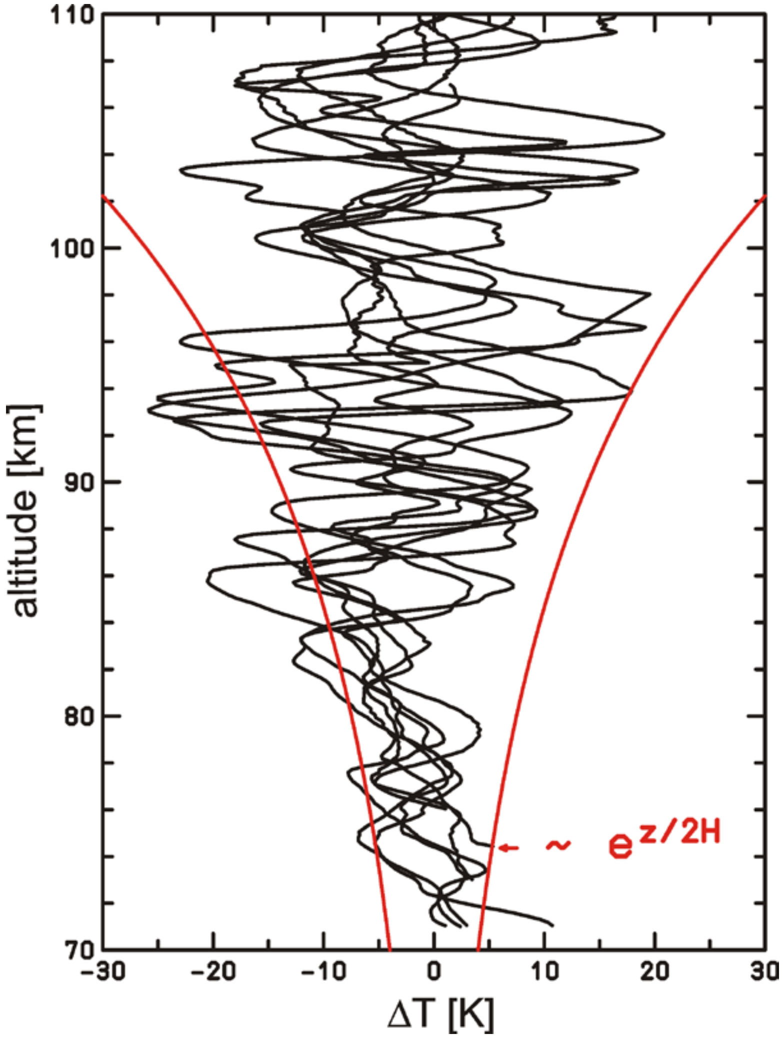
gravity waves

sound waves

GW polarization relations:

$$\begin{aligned} \tilde{u} &= -i\frac{m}{k}\frac{\omega}{N^2}\tilde{b} \\ \tilde{w} &= i\frac{\omega}{N^2}\tilde{b} \\ \tilde{\pi} &= -i\frac{\omega^2}{N^2}\frac{m}{c_p\overline{T}k^2}\tilde{b} \end{aligned}$$

GW breaking in the atmosphere



Rapp et al. (priv. comm.)

- Conservation

$$\frac{\rho_0}{2}\left(|\mathbf{v}|^2 + \ldots\right) \Rightarrow \boxed{\mathbf{v} \propto \frac{1}{\sqrt{\rho_0}}}$$

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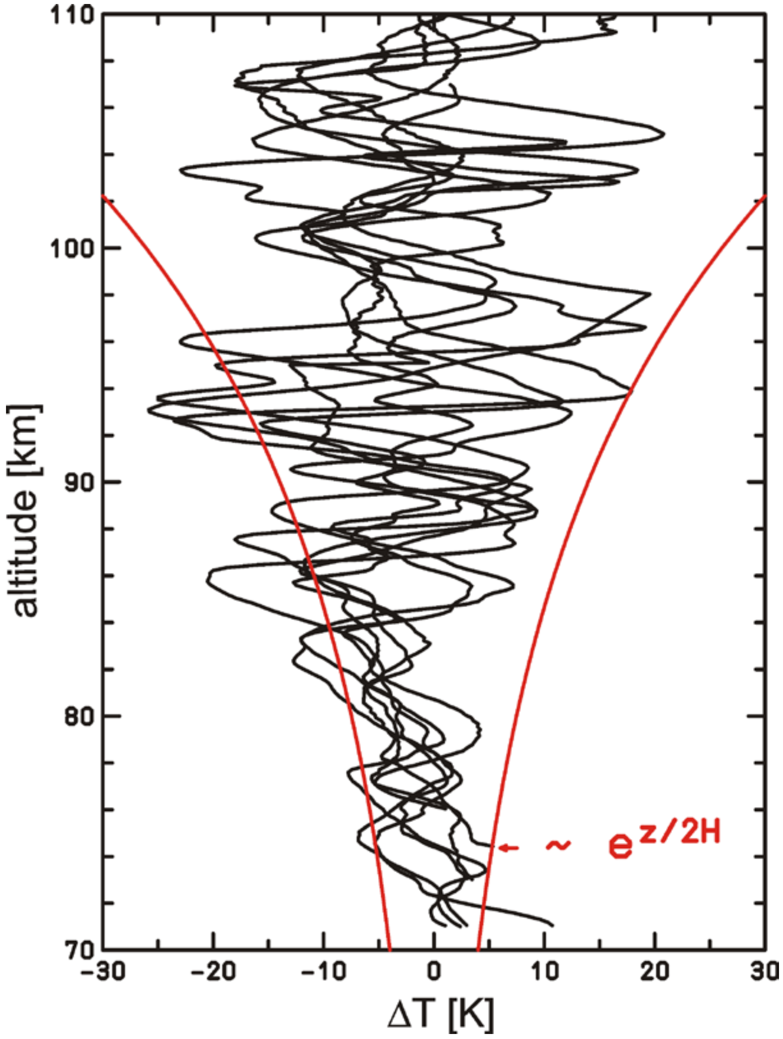

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- ***Instability*** at large altitudes

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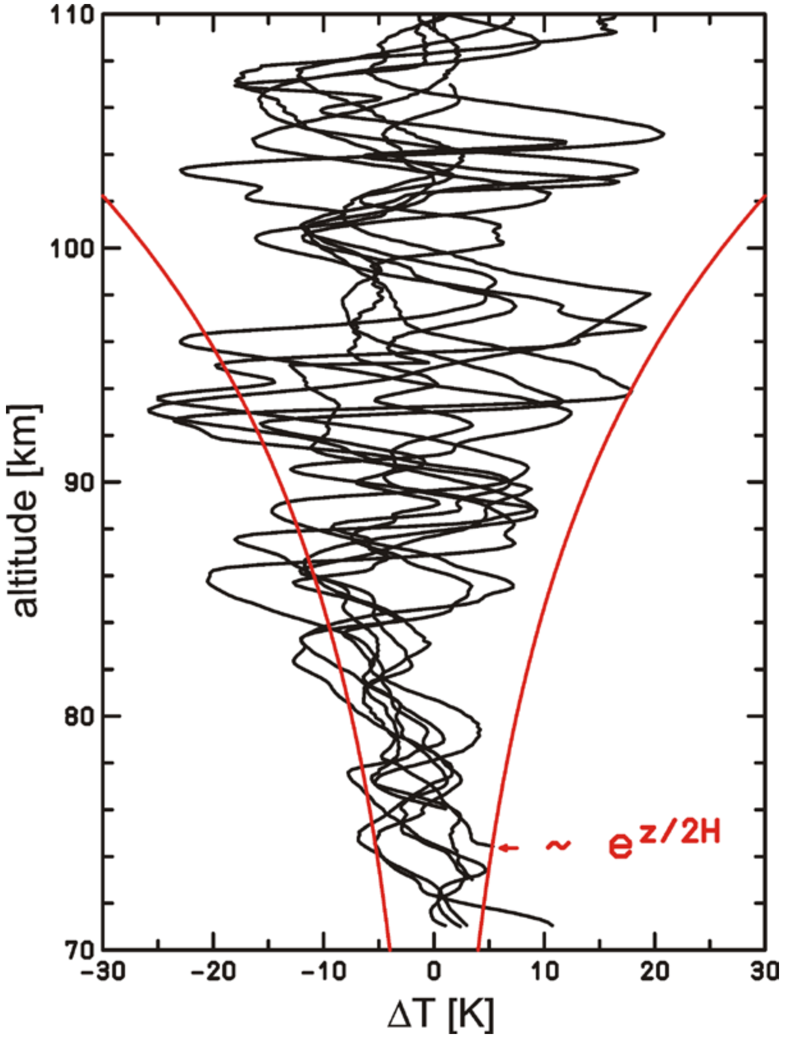

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- ***Instability*** at large altitudes
- **Turbulence**
- **Momentum deposition**…

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GW Parameterizations

- multitude of parameterization approaches
  (Lindzen 1981, Medvedev und Klaassen 1995, Hines 1997, Alexander and Dunkerton 1999, Warner and McIntyre 2001)

- too many free parameters

- reasons :
  - …
  - insufficent knowledge: ***conditions of wave breaking***

- basic paradigm: ***breaking of a single wave***
  - Stability analyses (NMs, SVs)
  - Direct numerical simulations (DNS)

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GW breaking

GW stability in the Boussinesq theory

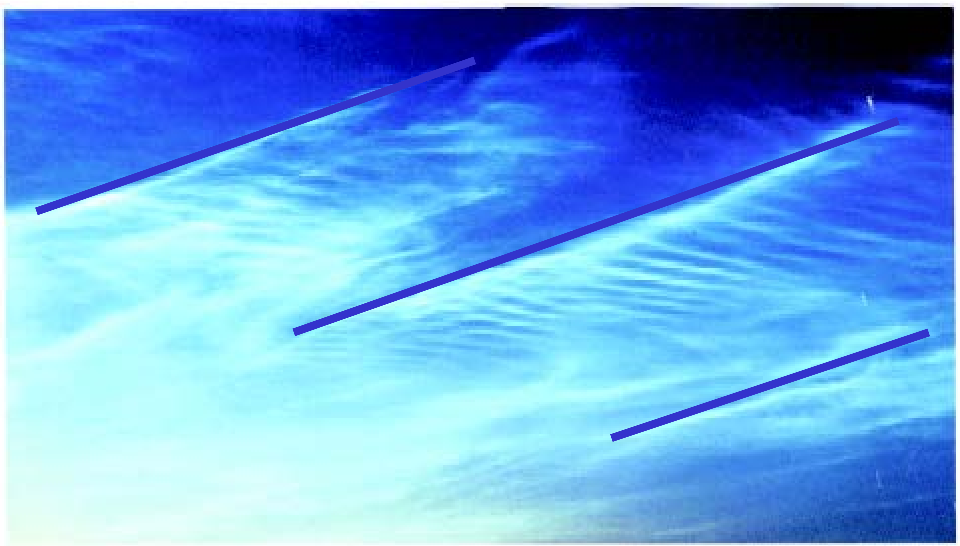
$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} + f\mathbf{k} \times \mathbf{v} + \nabla p - \mathbf{k}b = \nu\nabla^2\mathbf{v}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)b + N^2 w = \mu\nabla^2 b$$

$$\nabla \cdot \mathbf{v} = 0$$

$$b = g\frac{\theta - \overline{\theta}(z)}{\theta_0}, \qquad N^2 = \frac{g}{\theta_0}\frac{d\overline{\theta}}{dz}$$

Basic GW types



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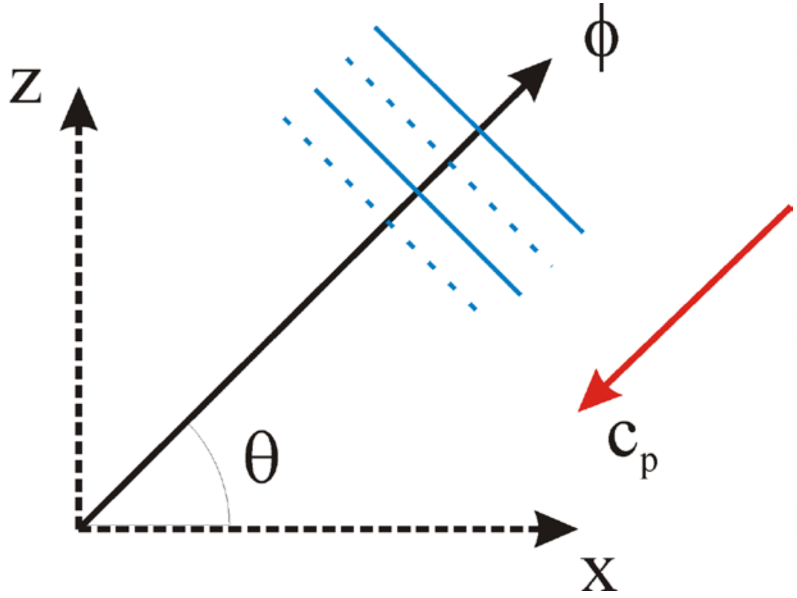

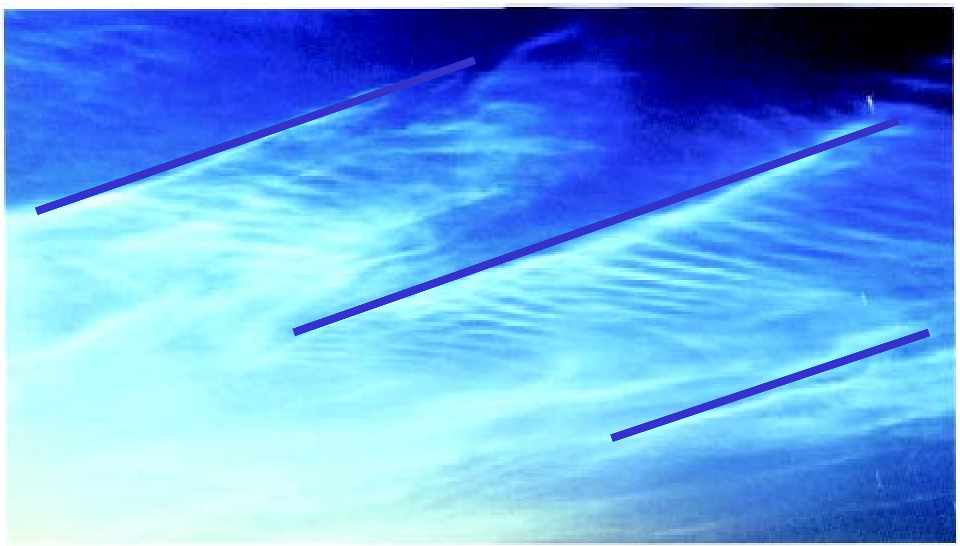

- Dispersion relation:

$$\omega = \pm\left(f^2 \sin^2\Theta + N^2 \cos^2\Theta\right)^{1/2}$$

$$f = 2\frac{2\pi}{24\text{h}}\sin\phi$$

Coriolis parameter

$$N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}$$

Stability

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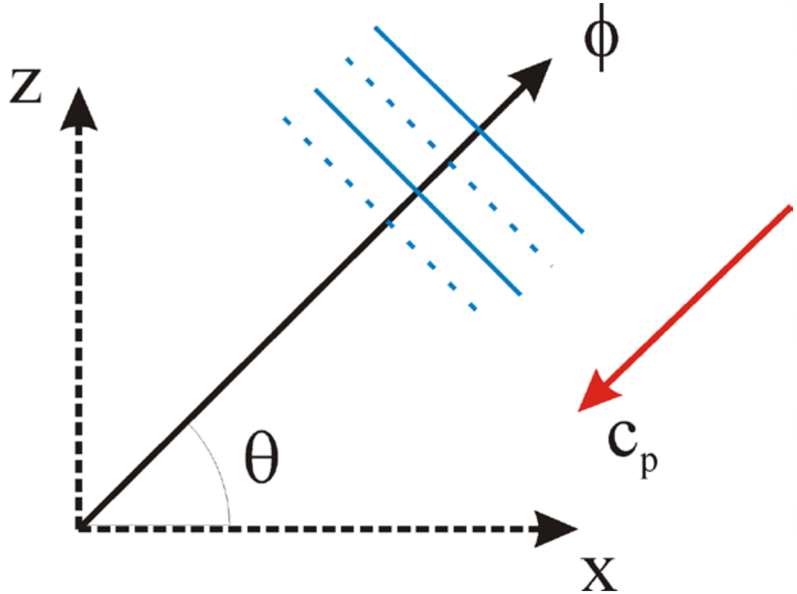

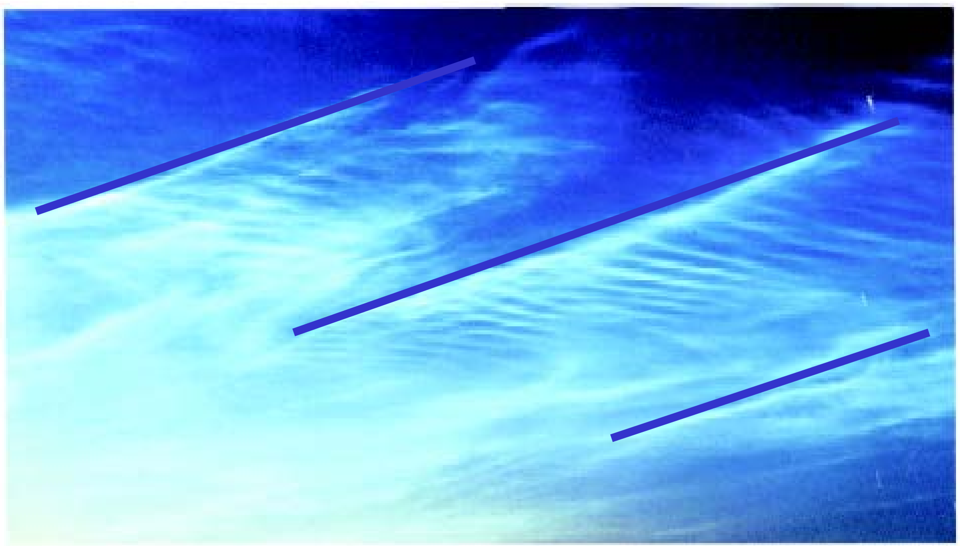

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Coriolis parameter

- ***Inertia-gravity waves (IGW):***

$$\Theta \approx 90^\circ \quad \Rightarrow \quad \omega \approx \pm f\left(1 + \frac{N^2}{2f^2}\cot^2\Theta\right)$$

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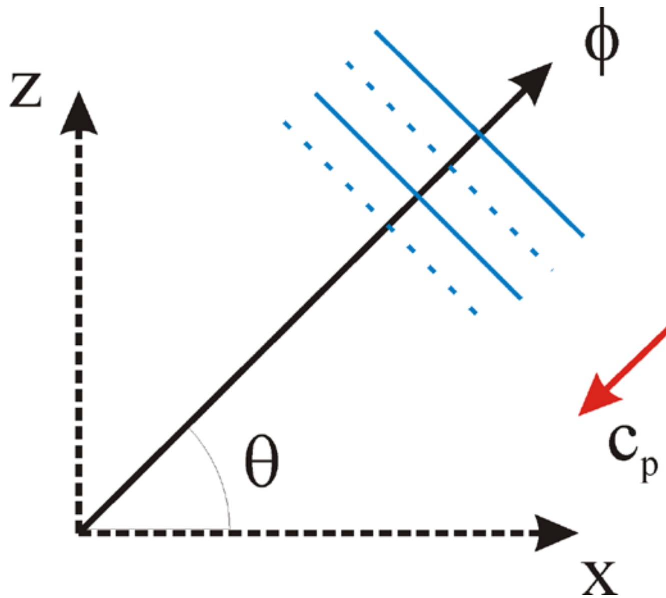

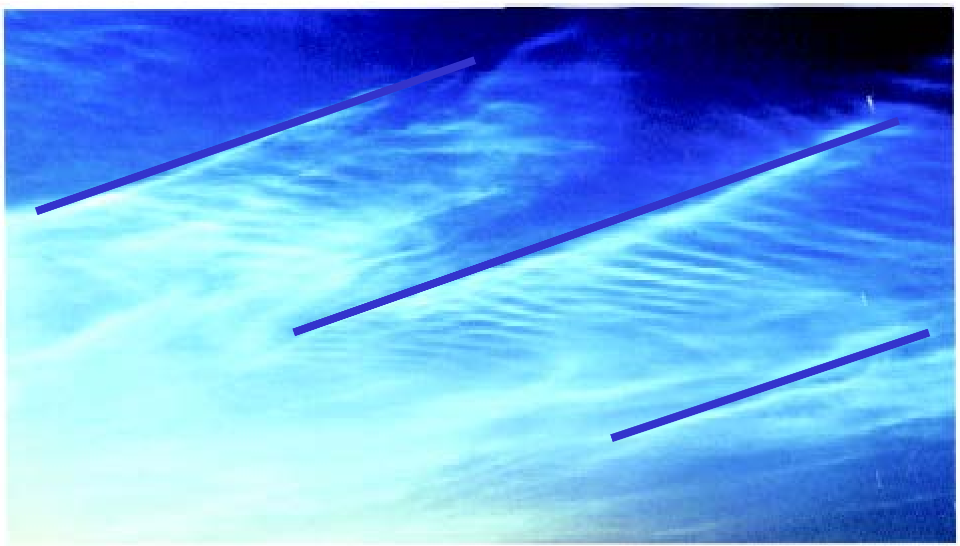

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Stability

- ***High-frequent GWs (HGW):***

$$\Theta < 90° \quad \Rightarrow \quad \omega \approx \pm N\cos\Theta$$

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Traditional Instability Concepts

- ***static (convective) instability:***

$$\frac{\partial B_{tot}}{\partial z} = N^2 + \frac{\partial b}{\partial z} < 0$$

amplitude reference (*a*>1)

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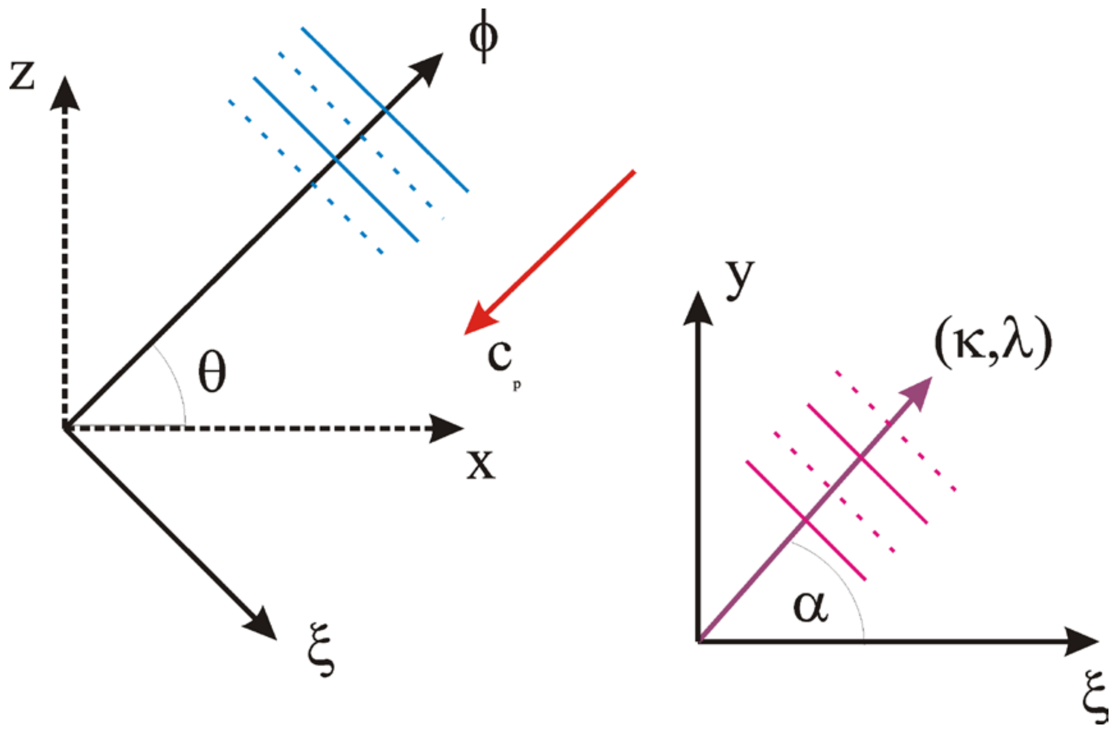
- ***BUT: limited applicability to GW breaking***

NM analysis, 2.5D-DNS



$$\begin{pmatrix}\mathbf{v}\\ b\end{pmatrix}(\xi,y,\phi,t=0)=\begin{pmatrix}\mathbf{v}\\ b\end{pmatrix}_{\mathrm{SW}}(\phi)+\begin{pmatrix}\mathbf{v}\\ b\end{pmatrix}_{\mathrm{NM,SV}}(\phi)\exp\left[i\underbrace{(\kappa\xi+\lambda y)}_{kx_{\parallel}}\right]$$

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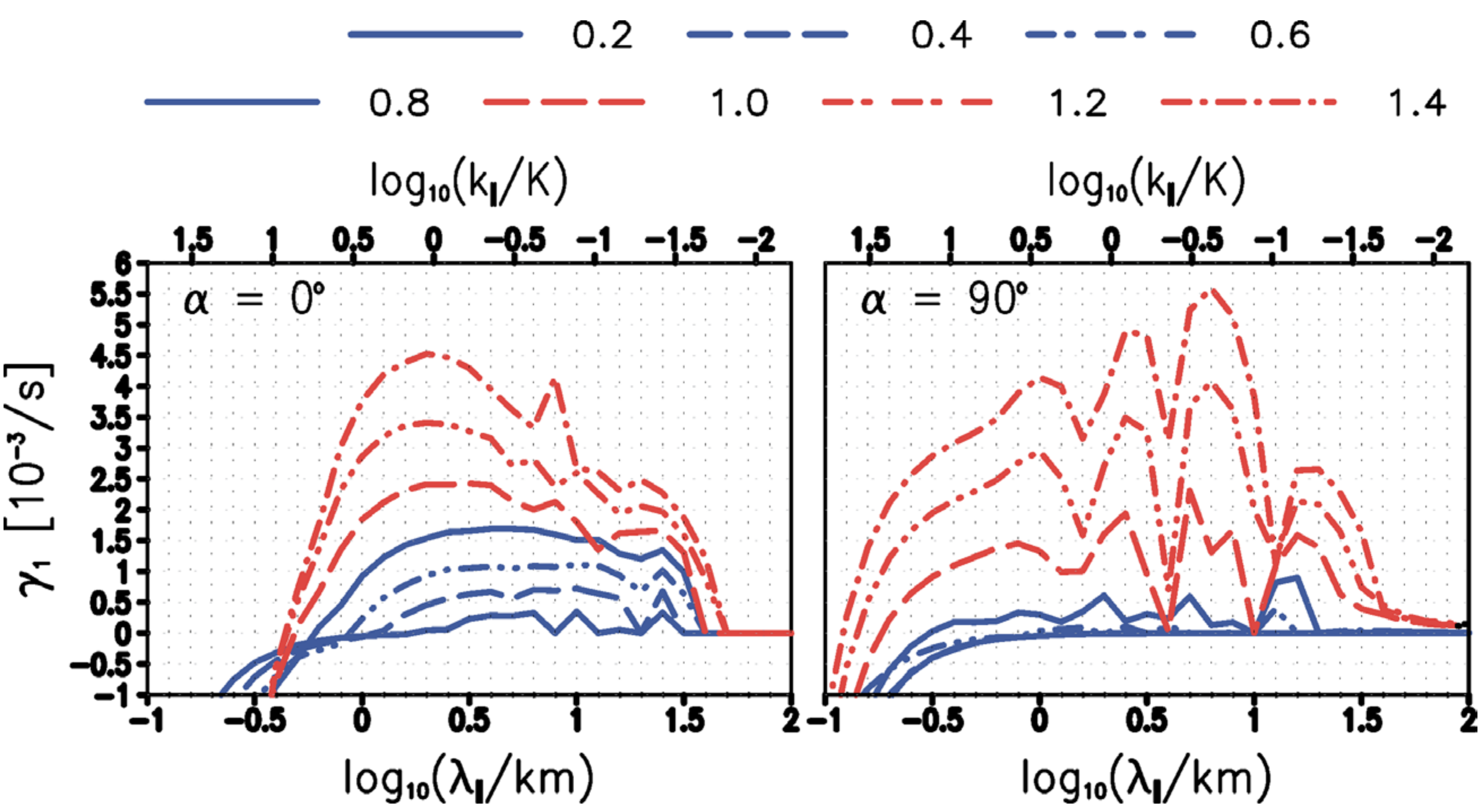
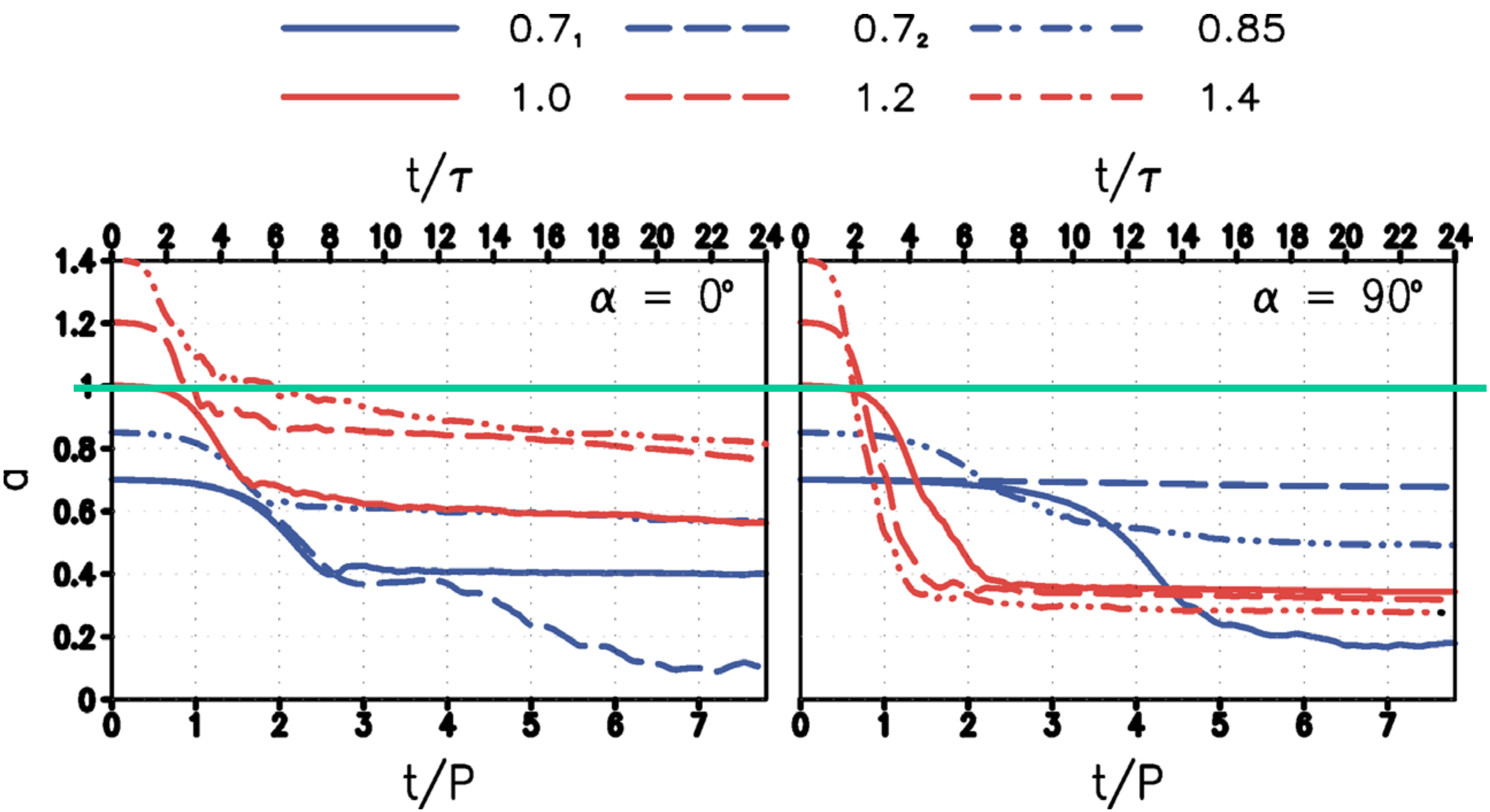
NMs: Growth Rates

($\Theta = 70°$)



Achatz (2005, 2007b)

GW amplitude after a perturbation by a NM (DNS):



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Energetics

- average over ξ and $y \rightarrow \left(\overline{\mathbf{v}}, \overline{b}\right)$
- deviation $\rightarrow \left(\mathbf{v}', b'\right)$

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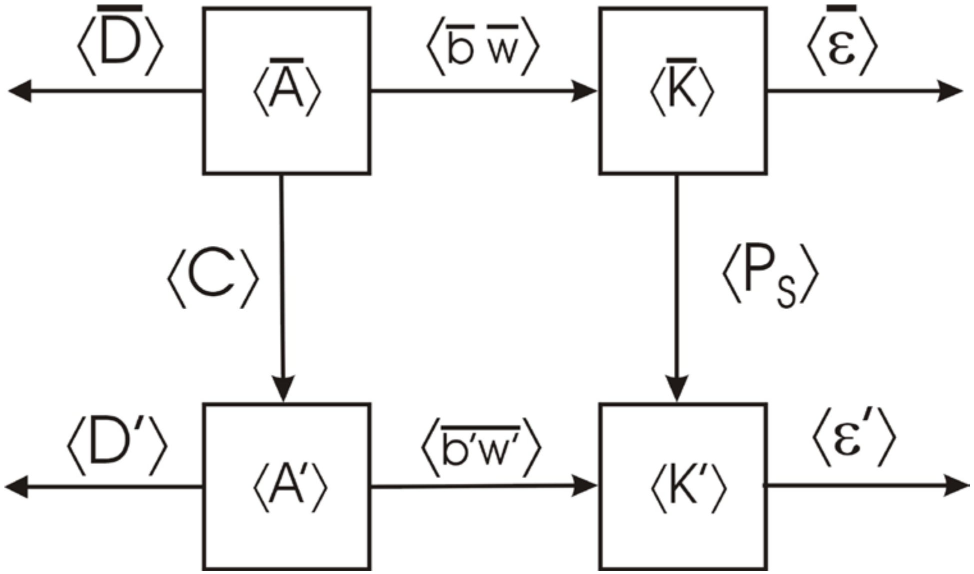

$$K = \frac{1}{2}|\mathbf{v}|^2 \qquad A = \frac{b^2}{2N^2}$$

$$P_S = -\overline{\mathbf{v}'u'_\phi} \cdot k\frac{d\overline{\mathbf{v}}}{d\phi} \qquad C = -\overline{b'u'_\phi}k\frac{d\overline{b}}{d\phi}$$

not the gradients in z matter,
but those in ϕ!

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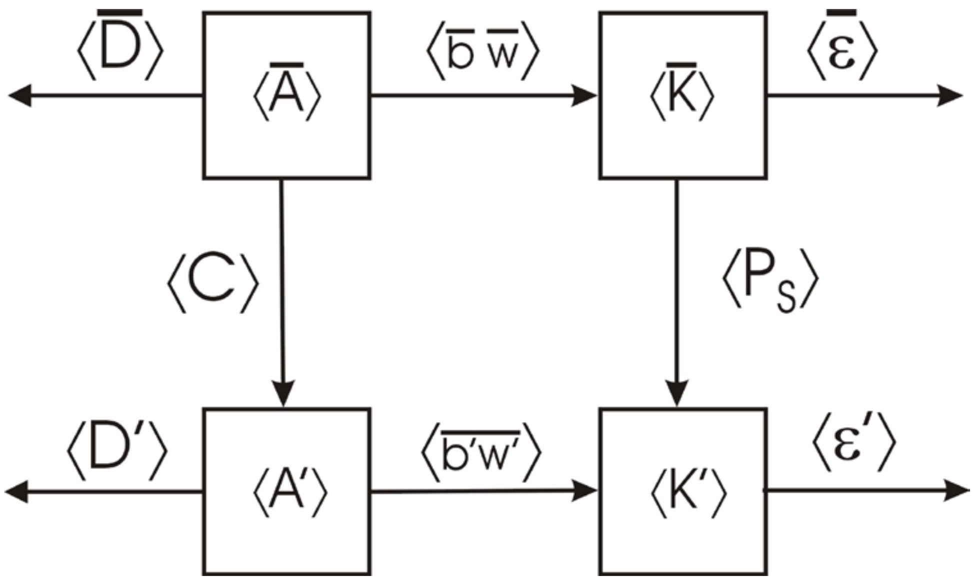

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instantaneous growth rate:

$$E' = K' + A' \Rightarrow \Gamma = \frac{d\langle E'\rangle / dt}{2\langle E'\rangle} = \Gamma_P + \Gamma_C + \Gamma_D + \Gamma_\varepsilon$$

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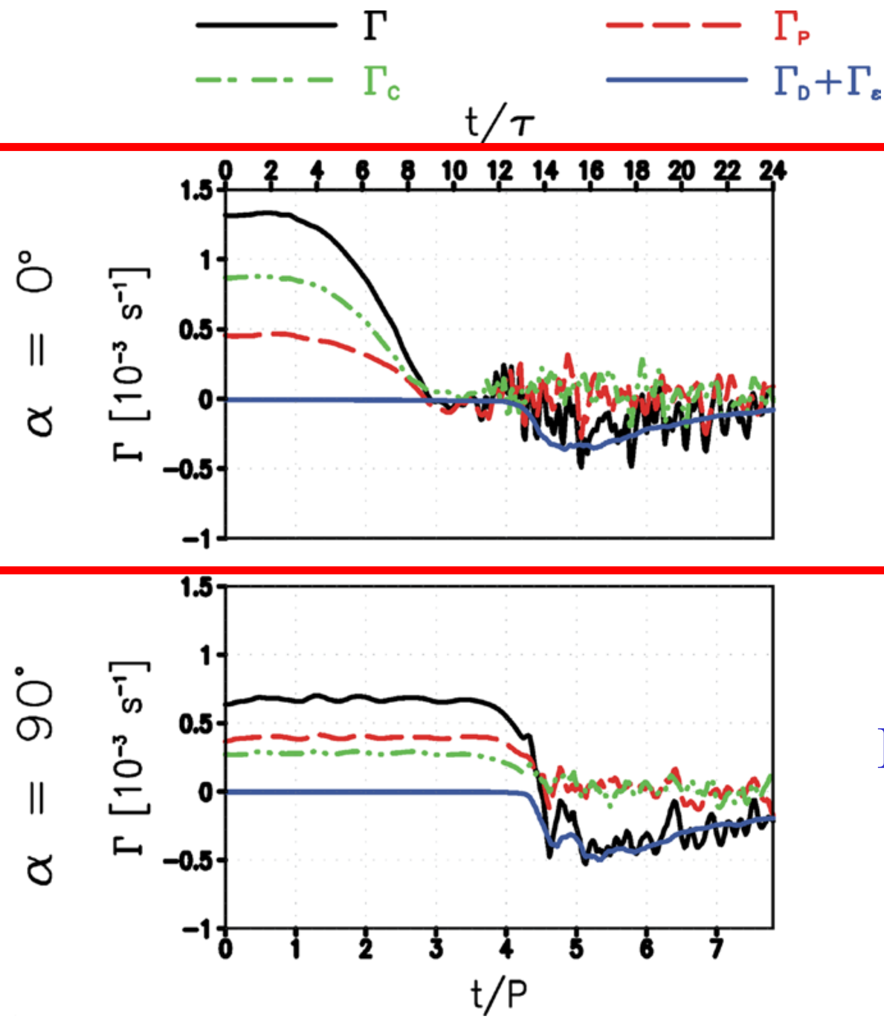
Energetics of a Breaking HGW with $a_0 < 1$



Strong growth perturbation energy

buoyancy instability

$$\Gamma = \frac{d\langle E' \rangle / dt}{2\langle E' \rangle} = \Gamma_P + \Gamma_C + \Gamma_D + \Gamma_\varepsilon$$

Achatz (2007b)

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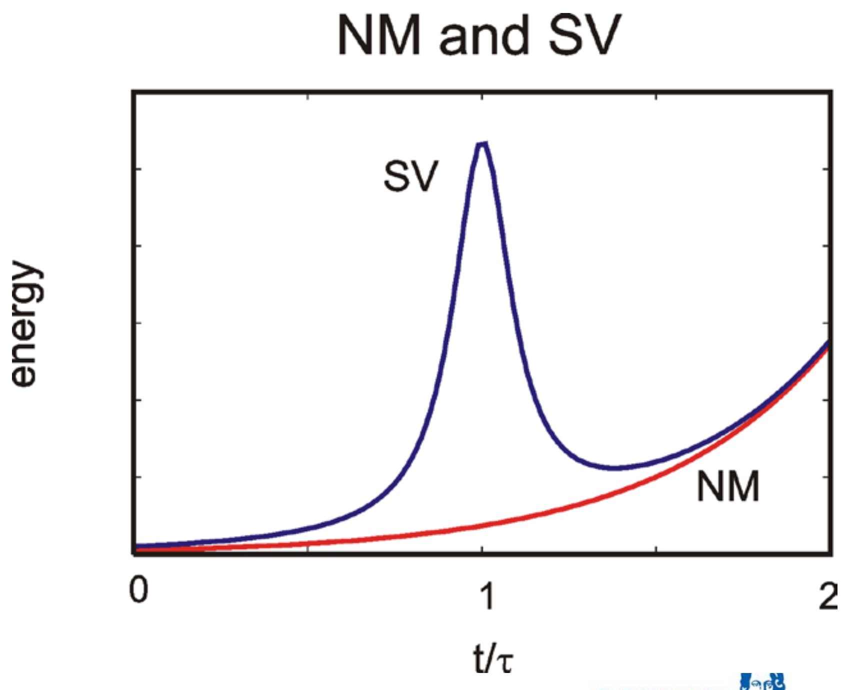
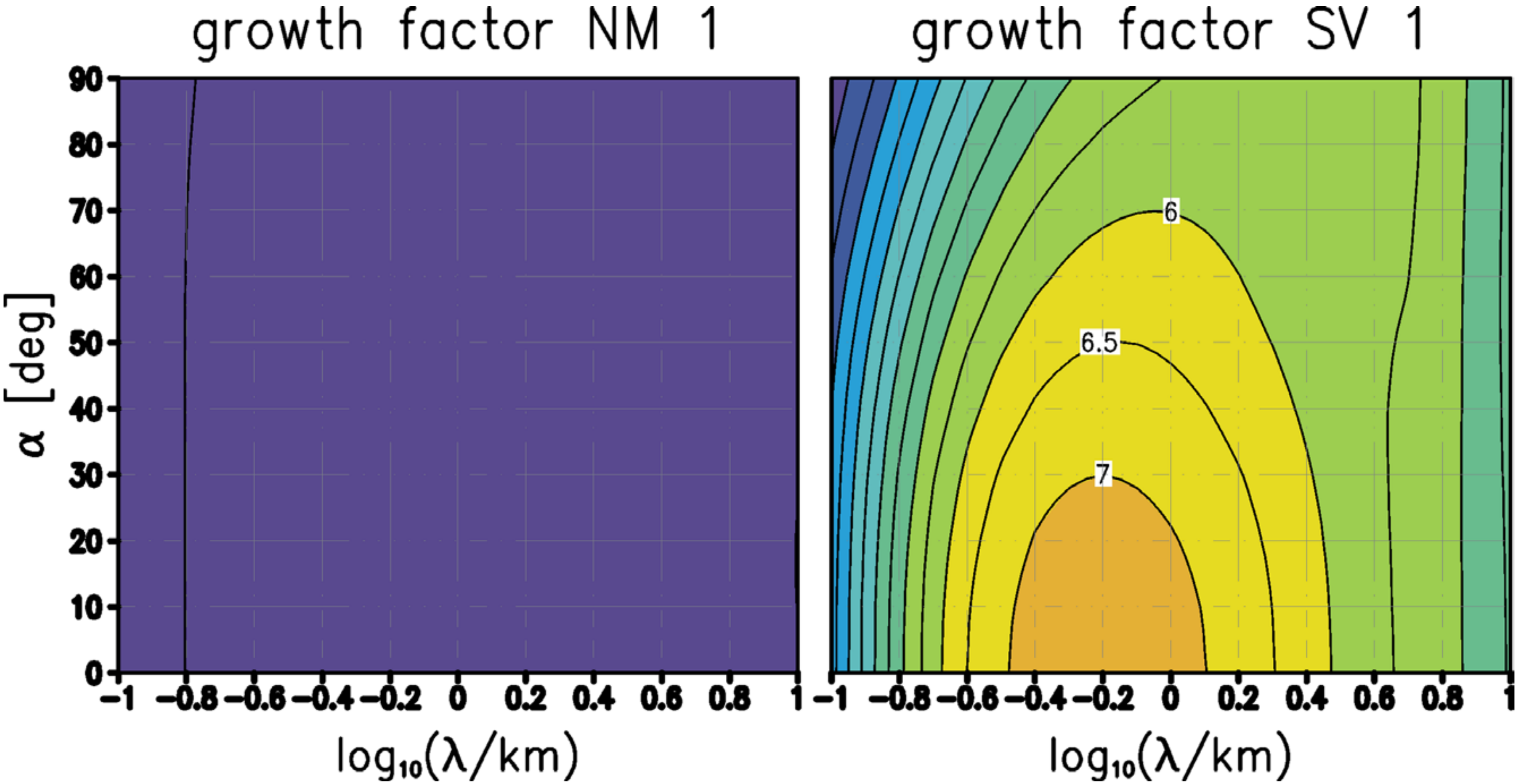
Singular Vectors:

characteristic perturbations in a linear initial value problem:

- ***normal modes***:
  Exponential energy growth
- ***singular vectors***:
  optimal growth over a finite time τ
  (Farrell 1988, Trefethen et al. 1993)

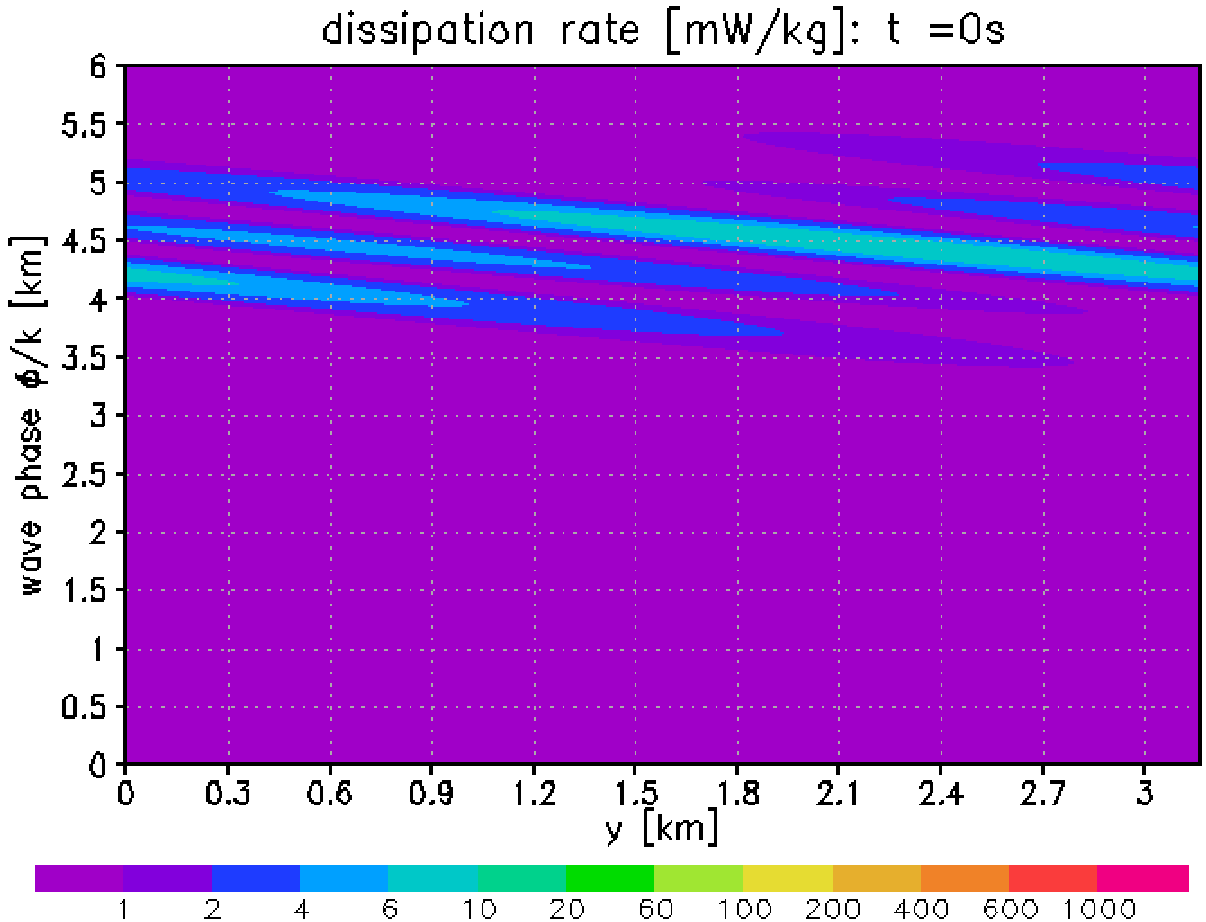


NMs and SVs of an IGW (Ri > ¼!): growth within 5min



Achatz (2005) Achatz and Schmitz (2006a,b)

Dissipation rate breaking IGW (a< 1, Ri > ¼):



Measured dissipation rates 1…1000 mW/kg (Lübken 1997, Müllemann et al. 2003)

Achatz (2007a)

Summary GW breaking

In comparison to assumptions in GW parameterizations:

- GW breaking sets in at ***lower amplitudes*** , i.e. it sets in ***earlier (at lower altitudes)***

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Summary GW breaking

In comparison to assumptions in GW parameterizations:

- GW breaking sets in at ***lower amplitudes*** , i.e. it sets in ***earlier (at lower altitudes)***

- GW ***dissipation stronger*** than assumed, i.e. ***more momentum deposition***

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Soundproof Modelling
and
Multi-Scale Asymptotics

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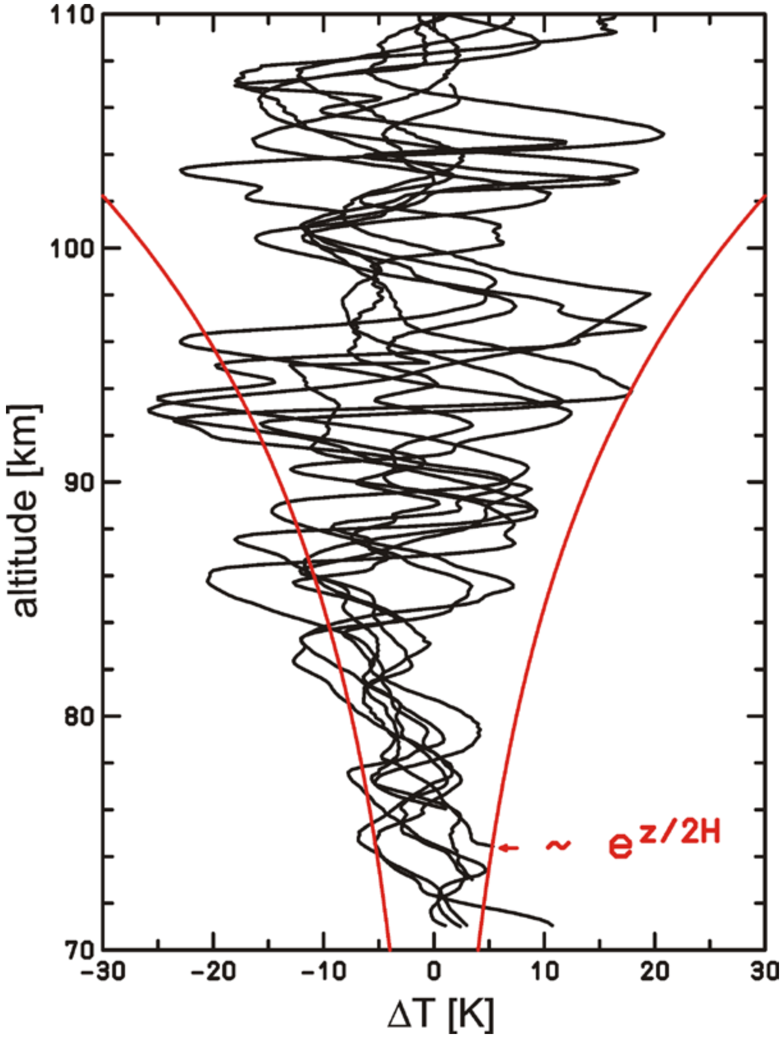
GW breaking in the middle atmosphere



Rapp et al. (priv. comm.)

- Conservation

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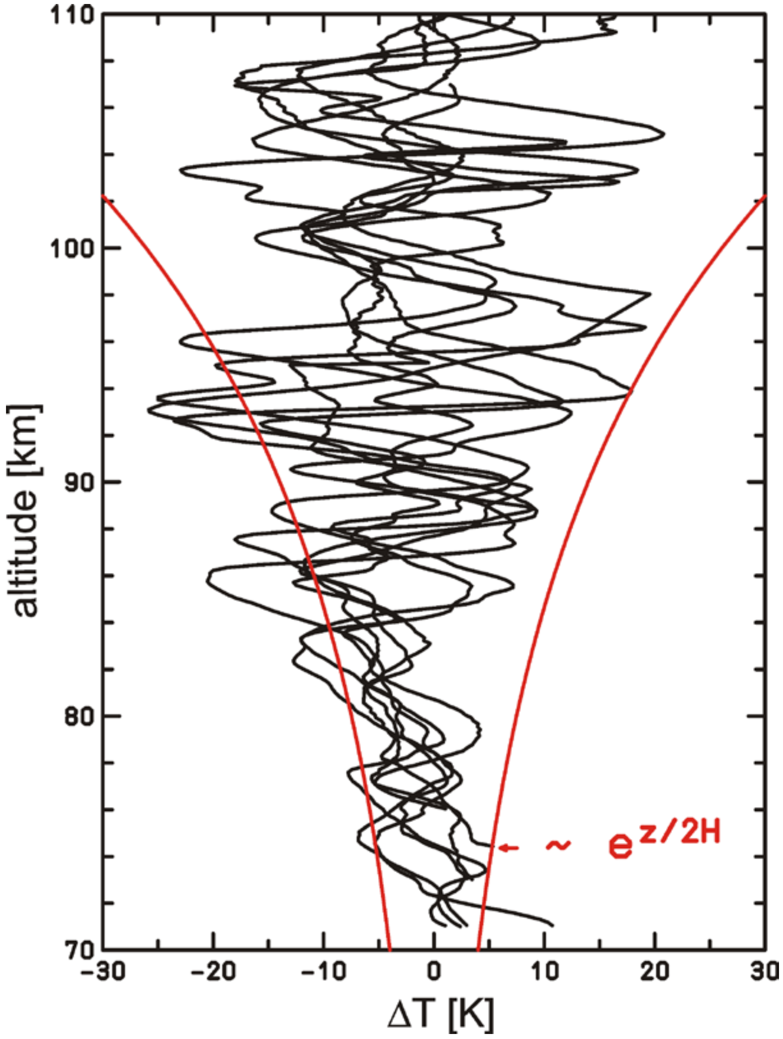

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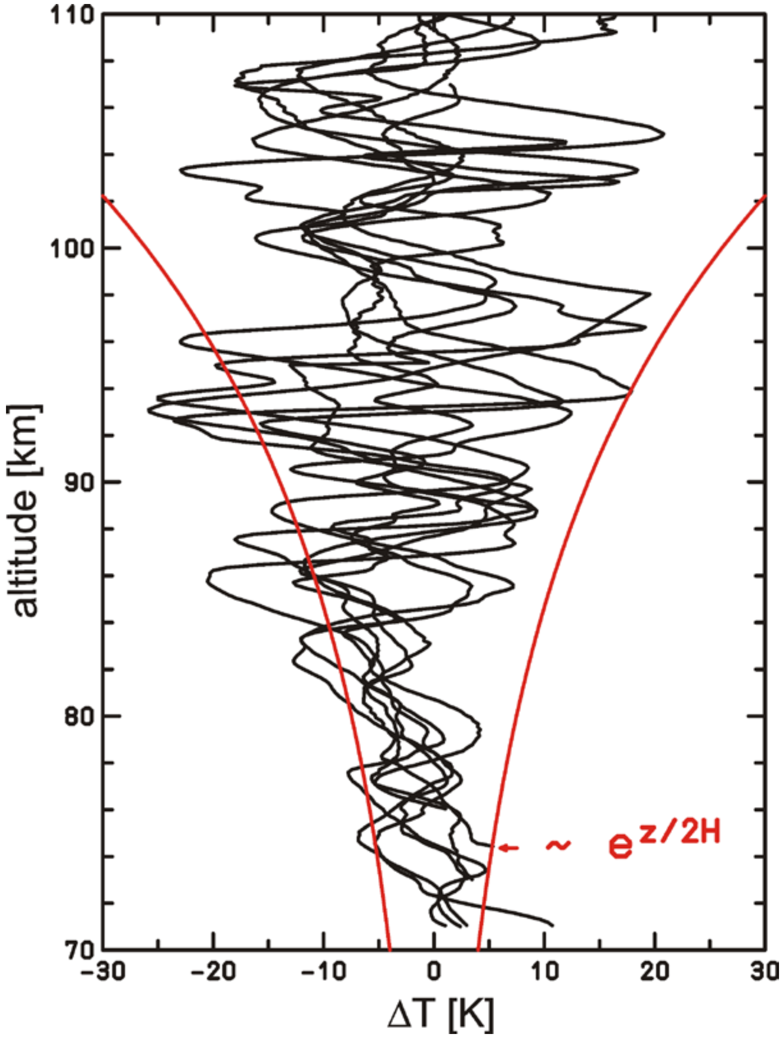

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  - *Anelastic (Ogura and Philips 1962, Lipps and Hemler 1982)*
  - *Pseudo-incompressible (Durran 1989)*

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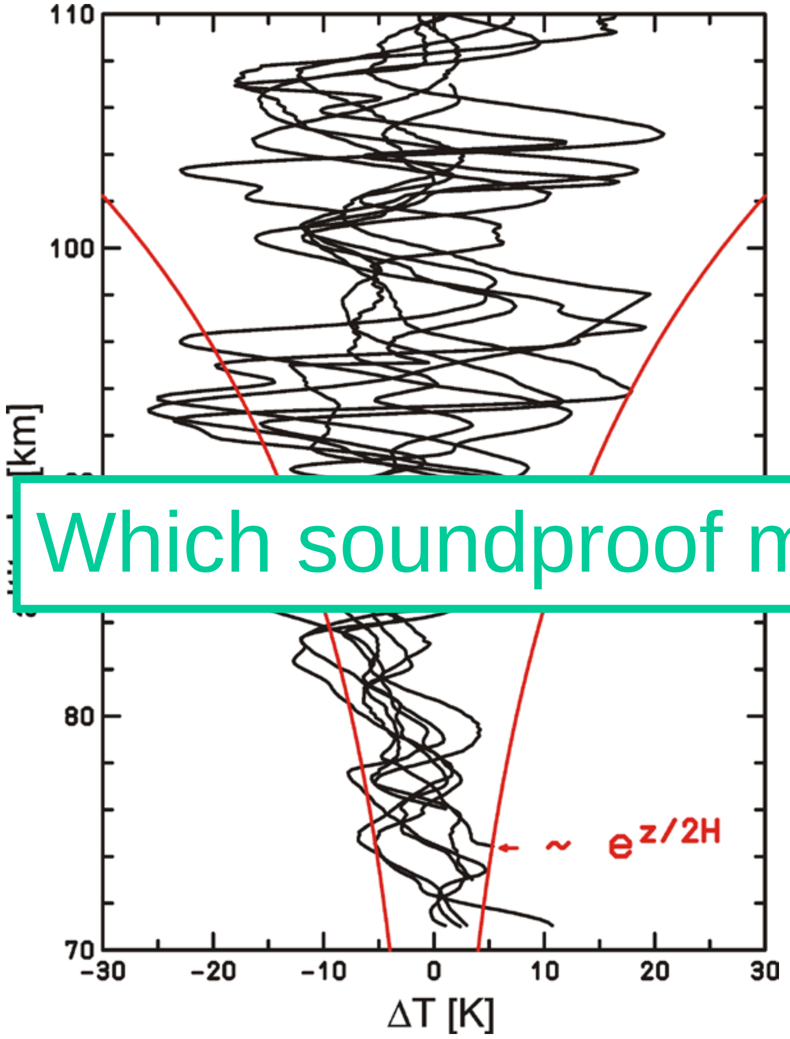

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growth and dissipation!

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Which soundproof model should be used?

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Scales: time and space

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$$t = T\hat{t} \qquad T = 1/\Omega = \text{ characteristic time scale}$$

dispersion relation for $K >> 1/2H$

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2 + \frac{1}{4H^2}} \approx \frac{N^2 k^2}{k^2 + m^2} \quad \Rightarrow \quad \boxed{\Omega = N = \frac{g}{\sqrt{c_p T_{00}}}}$$

Scales: velocities

- winds determined by polarization relations:

$$\tilde{u} = -i\frac{m}{k}\frac{\omega}{N^2}\tilde{b}$$

$$\tilde{w} = i\frac{\omega}{N^2}\tilde{b}$$

what is $\tilde{b}$?

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what is $\tilde{b}$?

- most interesting dynamics when GWs are <u>close to breaking</u>, i.e. locally

$$\frac{\partial \theta'}{\partial z} + \frac{d\overline{\theta}}{dz} = 0 \quad \Rightarrow \quad \frac{\partial b'}{\partial z} + N^2 w = 0$$

$$\Rightarrow$$

$$\left|\tilde{b}\right| = \frac{N^2}{|m|} \quad \Rightarrow \quad \mathbf{v} = U\hat{\mathbf{v}} \qquad U = \frac{\Omega}{K} = \frac{L}{T}$$

Non-dimensional Euler equations

- using:

$$\mathbf{x} = L\hat{\mathbf{x}}$$

$$t = T\hat{t}$$

$$\mathbf{v} = U\hat{\mathbf{v}}$$

$$\pi = \hat{\pi}$$

$$\theta = T_0\hat{\theta}$$

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yields

$$\varepsilon^2 \frac{D\hat{\mathbf{v}}}{D\hat{t}} = -\hat{\theta}\hat{\nabla}\hat{\pi} - \varepsilon\mathbf{e}_\mathbf{z}$$

$$\frac{D\hat{\pi}}{D\hat{t}} + \frac{\kappa}{1-\kappa}\hat{\pi}\hat{\nabla}\cdot\hat{\mathbf{v}} = 0$$

$$\frac{D\hat{\theta}}{D\hat{t}} = 0$$

$$\varepsilon = \frac{L}{H_\theta} << 1$$

$$H_\theta = \frac{c_p}{R} H = \frac{c_p T_{00}}{g}$$

isothermal potential-temperature scale height

Scales: thermodynamic wave fields

$$\left|\tilde{b}\right| = \frac{N^2}{|m|}$$

$$\Rightarrow$$

- potential temperature:

$$\frac{\theta'}{T_0} = \frac{\overline{\theta}}{T_0}\frac{\tilde{b}}{g} = O(\varepsilon)$$

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$$\Rightarrow$$

- potential temperature:

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- Exner pressure:

$$\tilde{\pi} = -i\frac{\omega^2}{N^2}\frac{m}{c_p\overline{T}k^2}\tilde{b} \quad \Rightarrow \quad \pi' = O\left(\varepsilon^2\right)$$

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Multi-Scale Asymptotics

Additional vertical scale needed:

- $\overline{\pi}$ and $\overline{\theta}$ have vertical scale $H_\theta = \frac{L}{\varepsilon}$

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Therefore: **Multi-scale-asymptotic ansatz**

$$\begin{pmatrix} \hat{\mathbf{v}} \\ \hat{\theta} \\ \hat{\pi} \end{pmatrix} = \sum_{i=0}^{\infty} \varepsilon^i \begin{pmatrix} \mathbf{v}^{(i)} \\ \theta^{(i)} \\ \pi^{(i)} \end{pmatrix} \left(\hat{\mathbf{x}}, \hat{t}, \varsigma\right) \qquad \varsigma = \varepsilon \hat{z}$$

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also assumed:

$$\hat{\nabla}\theta^{(0)} = 0$$
$$\pi^{(1)} = 0$$

Scale asymptotics Euler: results

$$\hat{\nabla}\pi^{(0)} = 0$$

$$\frac{\partial \theta^{(0)}}{\partial \hat{t}} = \frac{\partial \pi^{(0)}}{\partial \hat{t}} = 0$$

$$\frac{\partial \pi^{(0)}}{\partial \varsigma} = -\frac{1}{\theta^{(0)}}$$

Hydrostatic, large-scale, background

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Hydrostatic, large-scale, background

$$\frac{D_0 u^{(0)}}{D\hat{t}} + \theta^{(0)} \frac{\partial \pi^{(2)}}{\partial \hat{x}} = 0$$

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Momentum equations and entropy equations as in Boussinesq or anelastic theory

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Momentum equations and entropy equations as in Boussinesq or anelastic theory

$$\hat{\nabla}\cdot\mathbf{v}^{(0)} = 0$$

$$w^{(0)}\frac{\partial\pi^{(0)}}{\partial\varsigma} + \frac{\kappa}{1-\kappa}\pi^{(0)}\left(\hat{\nabla}\cdot\mathbf{v}^{(1)} + \frac{\partial w^{(0)}}{\partial\varsigma}\right) = 0$$

Exner-pressure equation:

- leading order incompressibility (Boussinesq)
- next order yields density effect on amplitude

Scale asymptotics: pseudo-incompressible equations

scale-asymptotic
analysis of

$$\frac{D\mathbf{v}}{Dt} = -c_p \theta \nabla \pi - \mathbf{e_z} g$$

$$\frac{D\theta}{Dt} = 0$$

$$\nabla \cdot \left(\overline{\rho} \overline{\theta} \mathbf{v} \right) = 0$$

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scale-asymptotic analysis of

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Scale asymptotics: pseudo-incompressible equations

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- same scale asymptotics as Euler
- such a result **cannot** be obtained from the anelastic equations

Scale asymptotics: anelastic equations

scale-asymptotic
analysis of

$$\frac{D\mathbf{v}}{Dt} = -\nabla\left(\frac{p'}{\overline{\rho}}\right) + \mathbf{e_z} b$$

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$$\nabla \cdot \left(\overline{\rho}\mathbf{v}\right) = 0$$

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$$+ \frac{\kappa}{1-\kappa} \pi^{(0)} \frac{w^{(0)}}{\theta^{(0)}} \frac{\partial \theta^{(0)}}{\partial \varsigma} = 0$$

anelastic equations only consistent if:

$$\frac{\kappa}{1-\kappa} \left| \frac{1}{\theta^{(0)}} \frac{\partial \theta^{(0)}}{\partial \varsigma} \right| << \left| \frac{1}{\pi^{(0)}} \frac{\partial \pi^{(0)}}{\partial \varsigma} \right|$$

i.e. **potential-temperature scale >> Exner-pressure scale**

Large-Amplitude WKB

Large-amplitude WKB

$$\hat{\mathbf{v}} = \tilde{\mathbf{v}}^{(0)}$$

$$\hat{\theta} = \hat{\theta}^{(0)} + \varepsilon \tilde{\theta}^{(1)}$$

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$$\hat{\mathbf{V}}^{(0)} = \hat{\mathbf{V}}_0^{(0)}(\underbrace{\varepsilon\hat{t}}_{\tau}, \underbrace{\varepsilon\hat{x}}_{\chi}, \underbrace{\varepsilon\hat{z}}_{\zeta}) + \Re\left\{\hat{\mathbf{V}}_1^{(0)}(\tau,\chi,\zeta)\exp\left[\frac{i}{\varepsilon}\varphi(\tau,\chi,\zeta)\right]\right\}$$

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Mean flow with only large-scale dependence

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Wavepacket with
- large-scale amplitude
- wavenumber and frequency with large-scale dependence

$$\mathbf{k} = \nabla_{(\chi,\zeta)} \varphi$$

$$\omega = \frac{\partial \varphi}{\partial \tau}$$

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Large-amplitude WKB

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Wave induced mean flow

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Harmonics of the wavepacket due to nonlinear interactions

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- collect equal powers in ε
- collect equal powers in $\exp(i\varphi/\varepsilon)$
- **no linearization**!

Large-amplitude WKB: leading order

$$i\mathbf{k}\cdot\hat{\mathbf{V}}_1^{(0)} = 0$$

$$\underbrace{\begin{pmatrix} -i\hat{\omega} & 0 & 0 & ik \\ 0 & -i\hat{\omega} & -N & im \\ 0 & N & -i\hat{\omega} & 0 \\ ik & im & 0 & 0 \end{pmatrix}}_{M\left(\hat{\omega},\mathbf{k}\right)} \begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(2)} \end{pmatrix} = 0$$

$\hat{\omega} = \omega - k\hat{U}_0^{(0)}$ intrinsic frequency

Large-amplitude WKB: leading order

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$$\hat{\omega} = \omega - k\hat{U}_0^{(0)} \qquad \text{intrinsic frequency}$$

$$\det(M) = 0 \Rightarrow$$

$$\hat{\omega}^2 = N^2 \frac{k^2}{k^2+m^2}$$

$$\begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(2)} \end{pmatrix} = \text{Nullvector of } M$$

dispersion relation and structure as from Boussinesq

Large-amplitude WKB: 1st order

$$M\left(\hat{\omega},\mathbf{k}\right)\begin{pmatrix} \hat{U}_1^{(1)} \\ \hat{W}_1^{(1)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(3)} \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ -\frac{\partial \hat{U}_1^{(0)}}{\partial \chi} - \frac{\partial \hat{W}_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{\hat{W}_1^{(0)}}{\hat{\pi}^{(0)}}\frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} \end{pmatrix}$$

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Solvability condition leads to wave-action conservation

$$\frac{\partial}{\partial\tau}\left(\frac{E'}{\hat{\omega}}\right)+\nabla_{(\chi,\zeta)}\cdot\left(\mathbf{c}_g\frac{E'}{\hat{\omega}}\right)=0$$

$$E'=\frac{\hat{\rho}^{(0)}}{2}\left(\frac{\left|\hat{\mathbf{V}}_1^{(0)}\right|^2}{2}+\frac{1}{2N^2}\left|\frac{\hat{\Theta}_1^{(0)}}{\hat{\theta}^{(0)}}\right|^2\right)\quad \text{wave energy}$$

$$\mathbf{c}_g=\left(\hat{U}_0^{(0)}+\frac{\partial\hat{\omega}}{\partial k},\frac{\partial\hat{\omega}}{\partial m}\right)\quad \text{group velocity}$$

(Bretherton 1966, Grimshaw 1975, Müller 1976)

Large-amplitude WKB: 1st order

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Pseudo-incompressible divergence needed!

Solvability condition leads to wave-action conservation

(Bretherton 1966, Grimshaw 1975, Müller 1976)

$$\frac{\partial}{\partial \tau}\left(\frac{E'}{\hat{\omega}}\right)+\nabla_{(\chi,\zeta)}\cdot\left(\mathbf{c}_g\frac{E'}{\hat{\omega}}\right)=0$$

$$E'=\frac{\hat{\rho}^{(0)}}{2}\left(\frac{\left|\hat{\mathbf{V}}_1^{(0)}\right|^2}{2}+\frac{1}{2N^2}\left|\frac{\hat{\Theta}_1^{(0)}}{\hat{\theta}^{(0)}}\right|^2\right) \quad \text{wave energy}$$

$$\mathbf{c}_g=\left(\hat{U}_0^{(0)}+\frac{\partial\hat{\omega}}{\partial k},\frac{\partial\hat{\omega}}{\partial m}\right) \quad \text{group velocity}$$

Large-amplitude WKB: 1st order, 2nd harmonics

$$\hat{\mathbf{V}}^{(1)} = \ldots + \Re \sum_{\alpha=1}^{\infty} \left\{ \hat{\mathbf{V}}_{\alpha}^{(1)}(\tau, \chi, \zeta) \exp\left[\frac{i}{\varepsilon} \alpha \varphi(\tau, \chi, \zeta) \right] \right\}$$

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$\alpha = 2:$

$$M\left(2\hat{\omega}, 2\mathbf{k}\right) \begin{pmatrix} \hat{U}_2^{(1)} \\ \hat{W}_2^{(1)} \\ \frac{1}{N} \frac{\hat{\Theta}_2^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)} \hat{\Pi}_2^{(3)} \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ \ldots \end{pmatrix} \Rightarrow$$

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2nd harmonics are slaved

Large-amplitude WKB: 1st order, higher harmonics

$$\hat{\mathbf{V}}^{(1)} = \ldots + \Re \sum_{\alpha=1}^{\infty} \left\{ \hat{\mathbf{V}}_{\alpha}^{(1)}(\tau, \chi, \zeta) \exp\left[\frac{i}{\varepsilon} \alpha \varphi(\tau, \chi, \zeta) \right] \right\}$$

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higher harmonics vanish
(nonlinearity is weak)

Large-amplitude WKB: mean flow

$$\tilde{\mathbf{v}}^{(0)} = \hat{\mathbf{V}}^{(0)} + \varepsilon \hat{\mathbf{V}}^{(1)} + o(\varepsilon) \quad \text{(e.g.)}$$

$$\hat{\mathbf{V}}^{(0)} = \hat{\mathbf{V}}_0^{(0)}(\tau, \chi, \zeta) + \dots$$

$$\hat{\mathbf{V}}^{(1)} = \hat{\mathbf{V}}_0^{(1)}(\tau, \chi, \zeta) + \dots$$

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$$\frac{\partial \hat{U}_0^{(0)}}{\partial \chi} = 0$$

Horizontal flow horizontally homogeneous (non-divergent)

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$$\frac{\partial \hat{U}_0^{(0)}}{\partial \chi} = 0$$

$$\hat{W}_0^{(0)} = 0$$

No zero-order vertical flow

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$$\frac{\partial \hat{U}_0^{(0)}}{\partial \chi} = 0$$

$$\hat{W}_0^{(0)} = 0$$

$$\hat{\Theta}_0^{(1)} = 0$$

No first-order potential temperature

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$$\frac{\partial \hat{U}_0^{(0)}}{\partial \tau} + \hat{\theta}^{(0)} \frac{\partial \hat{\Pi}_0^{(0)}}{\partial \chi} = -\nabla \cdot \hat{\mathbf{F}}_{GW}^{U}$$

- acceleration by GW-momentum-flux divergence

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- GWs induce lower-order potential temperature variations

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- p.-i. wave-correction not appearing in anelastic dynamics

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$$\hat{W}_0^{(1)} \frac{\partial \hat{\theta}^{(0)}}{\partial \zeta} = -\nabla \cdot \hat{\mathbf{F}}_{GW}^{\Theta} = 0$$

- Wave-induced lower-order vertical flow vanishes

Summary

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- **conclusion:**
  - use pseudo-incompressible equations for studies of GW dynamics with altitude-dependent amplitude

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