A Monge-Ampère enhancement for semi-Lagrangian methods

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We solve the Lagrangian form of the equations of motion using the NFT semi-Lagrangian algorithm embedded in the EULAG model for flows that satisfy the incompressibility constraint.

$$\mathbf{x}_0 = \mathbf{x}_i - \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau\,, \qquad (1)$$

$$\rho(\mathbf{x}_i,t) = \widehat{\mathrm{J}}\rho(\mathbf{x}_0,t_0)\,, \qquad (2)$$

$$\psi(\mathbf{x}_i,t) = \psi(\mathbf{x}_0,t_0) + \int_T R\,dt\,, \qquad (3)$$

Truncation errors in the evaluation of (1) induce errors in the flow Jacobian and so its compatibility with (2) is not ensured.

To enforce the compatibility of (1) with (2), we introduce a correction that has the form of the gradient of a scalar potential.

$$(\mathbf{x}_0)_C = \mathbf{x}_i - \Delta t\,(\tilde{\mathbf{v}} - \nabla\phi) \qquad \tilde{\mathbf{v}} \approx (\Delta t)^{-1}\int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau$$

For (2) to be satisfied in incompressible flows, one must then find the solution to

$$✷ \quad \det\left\{\frac{\partial(\mathbf{x}_0)_C}{\partial\mathbf{x}}\right\} = 1\,,$$

which turns out to be a form of Monge-Ampère equation (MAE). A Jacobian-Free Newton-Krylov method is used to find approximate solutions to ✷

Analysing the solution of the MAE in the vicinity of its stationary points reveals that it compensates, via the trajectory correction, the anomalous fluid contraction or expansion induced by flow rotation and deformation.

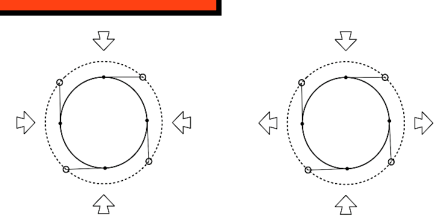


Fig. B: Direction of the applied MAE correction in the vicinity of a stationary point corresponding to pure flow rotation.

Semi-Lagrangian schemes form a class of numerical techniques that approximate the path integral (3) of the differential equation $d\psi/dt = R$ for the evolution of the intensive variable ψ (e.g. temperature, a velocity field component, etc...) with R symbolizing forcings on that variable (e.g. buoyancy, coriolis acceleration, Lorentz force, etc...). The path is defined by the trajectory integral (1) associated with the kinematic relation $d\mathbf{x}/dt = \mathbf{v}$.

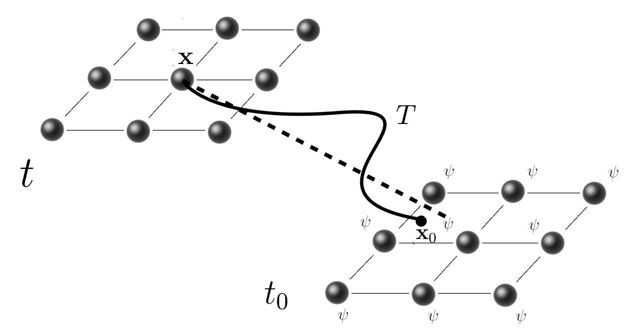


Fig. A: Solid and dashed lines denote the exact and approximate paths implied by equation (1) and to its discretized counterpart, respectively. Spheres denote points on the computational grid. Assuming the availability of ψ at the grid points at time t_0, advancing the solution in time invokes three distinct steps. First, the departure points $\mathbf{x}_0$ of the fluid particles arriving at $\mathbf{x}$ are evaluated by approximating (1). Second, the transported variable ψ is mapped to the set of the departure points by means of interpolation $\psi(\mathbf{x}_i,t_0) \rightarrow \psi(\mathbf{x}_0,t_0)$. Third, contributions from forcings are accounted for as integrals along the trajectories (3).

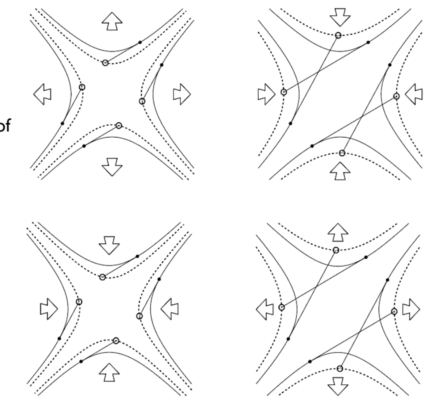


Fig. F: Direction of the applied MAE correction in the vicinity of a stationary point corresponding to pure flow deformation.

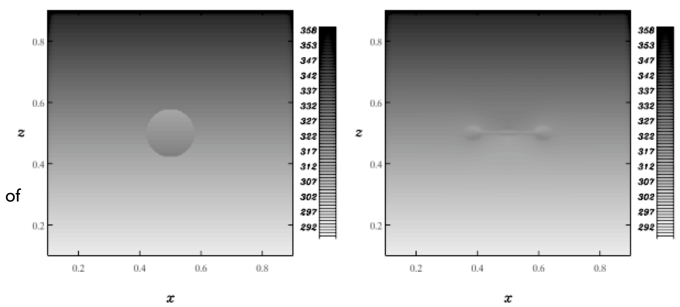


Fig. G: Coutour plots of the potential temperature distribution at initial time (left) and after 1.68 inverse Brunt-Väisälä frequencies.

Deformation and production of gravity waves takes place in the collapse of a stratified bubble of fluid inside an inviscid adiabatic Boussinesq flow.

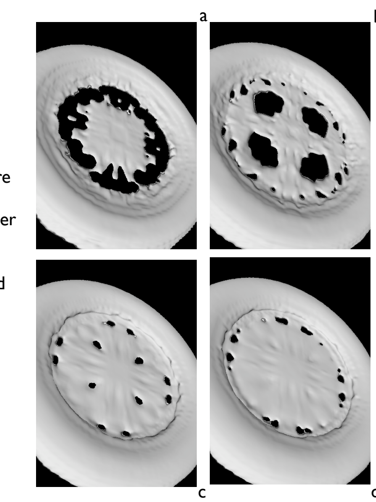


Fig. H: Isosurfaces of potential temperature resulting from first (top) and second order (bottom) trajectory evaluations for uncorrected (left) and corrected (right) SL schemes.

The topology of material elements is better preserved by the MA-enhanced SL scheme.

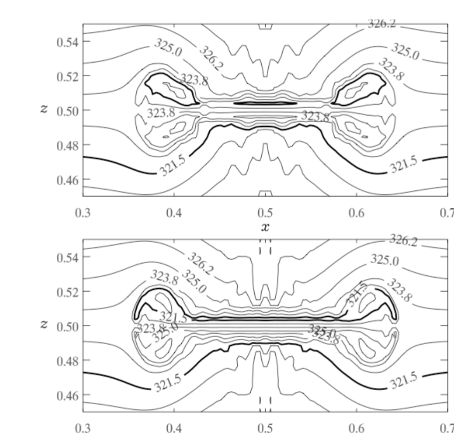


Fig. I: Cross-sections from figure H-a (top) and H-b (bottom). The regions corresponding to the bold level curve are multiply and simply connected for uncorrected and corrected SL schemes, respectively.

Enforcing compatibility amplifies the inverse energy cascade in 2D fully-developped turbulence.

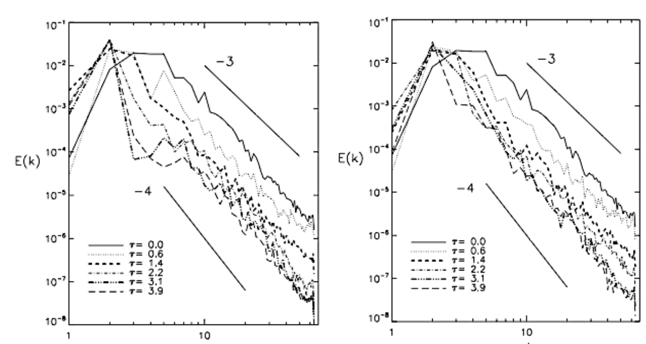
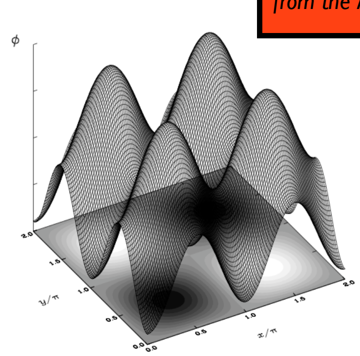


Fig. E: Time evolution of the power spectrum in a simulation of 2D decaying turbulence.

Fig. C: Graph of an approximate MAE solution in a pure rotational flow superposed over a contour plot of vorticity.



Enhanced shape preservation and scalar conservation result from the MAE corrrection.

Fig. D: *Top left*: Contour plot of the initial tracer. The dashed line shows the cross-section that is displayed in the other panels. *Top right*: Cross-section shown in the top left panel at various times for the uncorrected Euler-forward method. *Bottom left*: Initial tracer cross-section (solid line) and the uncorrected mid-point rule solution at t= 30 P (dashed line), *Bottom right*: Initial tracer cross-section (solid line) and MAE correction solution at t=30 P whwen applied to the second-order accurate scheme. P is the period of the trajectory passing through $x = \pi/2$.