# Unified linear time integration methods for compressible and anelastic models

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EULAG Workshop, Sopot

13.09.2010

LEIBNIZ-INSTITUT FÜR TROPOSPHÄRENFORSCHUNG

ASAM All Scale Atmospheric Model

MetStröm

- Motivation
- Different formulations of ASAM
- Rosenbrock-W-methods
- Rosenbrock-W-methods for index-2 problems
- Numerical examples
- Discussion

- Dry Euler equations in conservative form:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z} \\
\frac{\partial \rho u}{\partial t} &= -\frac{\partial \rho uu}{\partial x} - \frac{\partial \rho vu}{\partial y} - \frac{\partial \rho wu}{\partial z} - \frac{\partial p}{\partial x} \\
\frac{\partial \rho v}{\partial t} &= -\frac{\partial \rho uv}{\partial x} - \frac{\partial \rho vv}{\partial y} - \frac{\partial \rho wv}{\partial z} - \frac{\partial p}{\partial x} \\
\frac{\partial \rho w}{\partial t} &= -\frac{\partial \rho uw}{\partial x} - \frac{\partial \rho vw}{\partial y} - \frac{\partial \rho ww}{\partial z} - \frac{\partial p}{\partial z} - \rho g \\
\frac{\partial \rho \theta}{\partial t} &= -\frac{\partial \rho u\theta}{\partial x} - \frac{\partial \rho v\theta}{\partial y} - \frac{\partial \rho w\theta}{\partial z} + Q
\end{aligned}
$$

- Prognostic variables are total density $\rho$, momentum $\rho u$, $\rho v$, $\rho w$, and potential temperature times density $\rho\theta$. Pressure $p$ is a diagnostic variable from the equation of state

$$p = p_0 (R_d(\rho\theta)/p_0)^{1/(1-\kappa)}$$

with $\kappa = R_d/c_{pd}$ and $R_d$ gas constant of dry air and $c_{pd}$ specific heat of dry air at constant pressure.
- Pressure gradient $\nabla p = c_{pd} \rho\theta \nabla \pi$ with the Exner pressure $\pi$

- Anelastic approximation in conservative form
- Replace $\rho$ by $\rho_0 = \rho_0(z)$
- Cancel time derivative with respect to $\rho$
-

$$
\begin{aligned}
0 &= -\frac{\partial \rho_0 u}{\partial x} - \frac{\partial \rho_0 v}{\partial y} - \frac{\partial \rho_0 w}{\partial z} \\
\frac{\partial \rho_0 u}{\partial t} &= -\frac{\partial \rho_0 uu}{\partial x} - \frac{\partial \rho_0 vu}{\partial y} - \frac{\partial \rho_0 wu}{\partial z} - \rho_0 \frac{\partial \hat{\pi}}{\partial x} \\
\frac{\partial \rho_0 v}{\partial t} &= -\frac{\partial \rho_0 uv}{\partial x} - \frac{\partial \rho_0 vv}{\partial y} - \frac{\partial \rho_0 wv}{\partial z} - \rho_0 \frac{\partial \hat{\pi}}{\partial x} \\
\frac{\partial \rho_0 w}{\partial t} &= -\frac{\partial \rho_0 uw}{\partial x} - \frac{\partial \rho_0 vw}{\partial y} - \frac{\partial \rho_0 ww}{\partial z} - \rho_0 \frac{\partial \hat{\pi}}{\partial z} + \rho_0 \frac{\theta - \bar{\theta}}{\bar{\theta}} \\
\frac{\partial \rho_0 \theta}{\partial t} &= -\frac{\partial \rho_0 u\theta}{\partial x} - \frac{\partial \rho_0 v\theta}{\partial y} - \frac{\partial \rho_0 w\theta}{\partial z}
\end{aligned}
$$

- Prognostic variables are momentum $\rho_0 u$, $\rho_0 v$, $\rho_0 w$, and potential temperature times background density $\rho_0\theta$. Pressure $\hat{\pi}$ acts now as a Lagrange multiplier.

- Pseudo-incompressible approximation in conservative form
- Replace $\rho$ by $\rho = \frac{\rho_0\theta_0}{\theta}$.
- Cancel time derivative with respect to $\rho\theta$
- $$\frac{\partial\rho}{\partial t} = -\frac{\partial\rho u}{\partial x} - \frac{\partial\rho v}{\partial y} - \frac{\partial\rho w}{\partial z}$$
$$\frac{\partial\rho u}{\partial t} = -\frac{\partial\rho uu}{\partial x} - \frac{\partial\rho vu}{\partial y} - \frac{\partial\rho wu}{\partial z} - \rho_0\theta_0\frac{\partial\pi}{\partial x}$$
$$\frac{\partial\rho v}{\partial t} = -\frac{\partial\rho uv}{\partial x} - \frac{\partial\rho vv}{\partial y} - \frac{\partial\rho wv}{\partial z} - \rho_0\theta_0\frac{\partial\pi}{\partial x}$$
$$\frac{\partial\rho w}{\partial t} = -\frac{\partial\rho uw}{\partial x} - \frac{\partial\rho vw}{\partial y} - \frac{\partial\rho ww}{\partial z} - \rho_0\theta_0\frac{\partial\pi}{\partial z} - \rho g$$
$$0 = -\frac{\partial\rho_0 u\theta_0}{\partial x} - \frac{\partial\rho_0 v\theta_0}{\partial y} - \frac{\partial\rho_0 w\theta_0}{\partial z} + Q$$
- Prognostic variables are density $\rho$, momentum $\rho u$, $\rho v$, $\rho w$. Pressure $\pi$ acts now as a Lagrange multiplier.

- Last equation can be rewritten as
- $$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z} \\
\frac{\partial \rho u}{\partial t} &= -\frac{\partial \rho uu}{\partial x} - \frac{\partial \rho vu}{\partial y} - \frac{\partial \rho wu}{\partial z} - \rho_0\theta_0\frac{\partial \pi}{\partial x} \\
\frac{\partial \rho v}{\partial t} &= -\frac{\partial \rho uv}{\partial x} - \frac{\partial \rho vv}{\partial y} - \frac{\partial \rho wv}{\partial z} - \rho_0\theta_0\frac{\partial \pi}{\partial x} \\
\frac{\partial \rho w}{\partial t} &= -\frac{\partial \rho uw}{\partial x} - \frac{\partial \rho vw}{\partial y} - \frac{\partial \rho ww}{\partial z} - \rho_0\theta_0\frac{\partial \pi}{\partial z} - \rho g \\
0 &= -\frac{\partial \rho u \frac{\rho_0\theta_0}{\rho}}{\partial x} - \frac{\partial \rho v \frac{\rho_0\theta_0}{\rho}}{\partial y} - \frac{\partial \rho w \frac{\rho_0\theta_0}{\rho}}{\partial z} + Q
\end{aligned}$$
- Last equation is nonlinear in momentum and density
- Equation looks like a variable density flow

- Method of lines approach, spatial approximation leads to the time integration
-
$$\dot{y} = F(y)$$
-
$$\begin{aligned} Y_i &= y_n + \sum_{j=1}^{i-1} a_{ij} k_j \\ (I - \gamma_0 \Delta t_n W) k_i &= \Delta t_n F(Y_i) + \sum_{j=1}^{i} \gamma_{ij} k_j, \quad i = 1, \dots, s \\ Y_{n+1} &= y_n + \sum_{i=1}^{s} b_i k_i \end{aligned}$$
  where $s$ is the number of stages, $\gamma_0$, $a_{ij}$, $\gamma_{ij}$, $b_i$ parameters
- Matrix $W \approx \partial F(y_n)/\partial y$
- Convergence order does not depend on $W$
- If $W$ is identical zero a explicit Runge-Kutta is obtained

- $y = (u, p)^T$
-
$$\begin{aligned} \dot{u} &= F(u, p) \\ 0 &= G(u) \end{aligned}$$
- DAE is of index two if matrix $\frac{\partial G}{\partial u} \frac{\partial F}{\partial p}$ is regular

- To derive an Rosenbrock method for the DAE consider instead

$$\begin{aligned} \dot{u} &= F(u,p) \\ \dot{p} &= \frac{1}{\epsilon} G(u) \end{aligned}$$

-

$$\begin{aligned} U_i &= u_n + \sum_{j=1}^{i-1} a_{ij} k_j \\ P_i &= p_n + \sum_{j=1}^{i-1} a_{ij} l_j \\ \begin{pmatrix} I - \gamma_0 \Delta t_n W_{11} & -\gamma_0 \Delta t_n W_{12} \\ -\frac{1}{\epsilon}\gamma_0 \Delta t_n W_{21} & I \end{pmatrix} \begin{pmatrix} k_i \\ l_i \end{pmatrix} &= \begin{pmatrix} \Delta t_n F(U_i, P_i) + \sum_{j=1}^{i-1} \gamma_{ij} k_j \\ \frac{1}{\epsilon} \Delta t_n G(U_i) + \sum_{j=1}^{i-1} \gamma_{ij} l_j \end{pmatrix}, \quad i = \\ u_{n+1} &= u_n + \sum_{i=1}^{s} b_i k_i \\ p_{n+1} &= p_n + \sum_{i=1}^{s} b_i l_i \end{aligned}$$

where $W_{11} \approx \partial F(u_n, p_n)/\partial u$, $W_{12} = \partial F(u_n, p_n)/\partial p$, $W_{21} = \partial G(u_n)/\partial u$

- Multiplying by $\epsilon$ and $\epsilon \longrightarrow 0$
-

$$
\begin{aligned}
U_i &= u_n + \sum_{j=1}^{i-1} a_{ij} k_j \\
P_i &= p_n + \sum_{j=1}^{i-1} a_{ij} l_j \\
\begin{pmatrix} I - \gamma_0 \Delta t_n W_{11} & -\gamma_0 \Delta t_n W_{12} \\ -\gamma_0 \Delta t_n W_{21} & 0 \end{pmatrix} \begin{pmatrix} k_i \\ l_i \end{pmatrix} &= \begin{pmatrix} \Delta t_n F(U_i, P_i) + \sum_{j=1}^{i-1} \gamma_{ij} k_j \\ \Delta t_n G(U_i) \end{pmatrix}, \quad i = \\
u_{n+1} &= u_n + \sum_{i=1}^{s} b_i k_i \\
p_{n+1} &= p_n + \sum_{i=1}^{s} b_i l_i
\end{aligned}
$$

- With a suitable approximation to $W$ in the compressible case almost same linear algebra

- To reduce further computational cost the following approximation to the Jacobian are applied
- The Jacobian with respect to advection is computed for a first order approximation in space
- For a generic variable $\chi$ and advection in $x$-direction

$$\frac{\partial(\rho u)\frac{\rho\chi}{\rho}}{\partial x}$$

  the differentiation with respect to $\rho$ is ignored
- $(I - \gamma_0 \Delta t W)$ is replaced by $(I - \gamma_0 \Delta t W_T)(I - \gamma_0 \Delta t W_S)$ with $W = W_T + W_S$
- $W_T$ is an approximation to the transport part, $W_S$ is an approximation to the sound part
- Further simplifications are possible for special applications

- Sound part means differentiation of the pressure in the momentum equations with respect to the thermodynamic variables, differentiation of the right hand side of the thermodynamic variables with respect to momentum
- $$W_S = \begin{pmatrix} D_u & D_{u1}\,\mathrm{Grad}\,D_1 & \dots & D_{us}\,\mathrm{Grad}\,D_s \\ \mathrm{Div}\,T_{1u} & T_1 & \dots & 0 \\ \mathrm{Div}\,T_{su} & 0 & 0 & T_s \end{pmatrix}$$

  where $D$ and $T$ are diagonal matrices
- In the anelastic cases $T_i = 0$
- Choice of the diagonal matrices for the different formulations

| | $D_{u1}$ | $D_1$ | $T_{1u}$ |
|---|---|---|---|
| C | 1 | $\partial p/\partial\rho\theta$ | $\theta$ |
| A | $\rho_0$ | 1 | 1 |
| P | $\rho_0\theta_0$ | 1 | $(\rho_0\theta_0)/\rho$ |

- Underlying explicit Runge-Kutta method is the RK3 method

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 1/2 & 0 & 1/2 & \\ 1 & 0 & 0 & 1 \end{array}$$

- Used in the codes WRF, NICAM and COSMO
- $\gamma_0 = 1$
- Coefficients $a$ and $b$

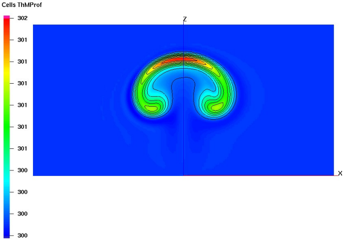
| | | |
|---|---|---|
| 0.3333333333333331 | | |
| 0.2037037037037036 9 | 0.5000000000000000 | |
| -0.6296296296296296 5 | 2.7500000000000000 | 1.0000000000000000 |

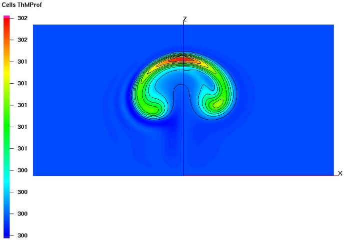
- Coefficients $\gamma$

| | |
|---|---|
| -0.4074074074074073 8 | |
| 0.6296296296296296 5 | -2.7500000000000000 |

- Other Rosenbrock-W-methods are available (c.f. Rang and Angermann)

- Translating bubble, left RosRK3, right Ro3PW (Rang, Angermann)



- Method is implemented in the code ASAM (All Scale Atmospheric Model)
- Cut cell approach
- Cartesian or spherical grid
- Block structured for AMR and parallelization
- Different time integration methods
- asam.tropos.de

- Domain 20.000 m by 10.000 m
- $\Delta x = \Delta z = 125\text{m}$, $\Delta t = 5\text{s}$
- Initial conditions for the unperturbed environment (calm, hydrostatic, neutrally stable)
- Perturbation

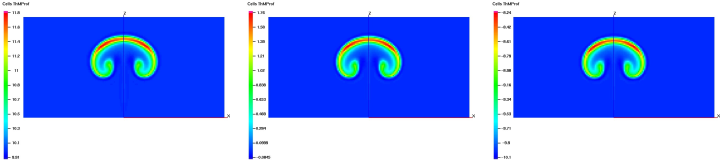
$$\Delta\theta = \begin{cases} 0 & \text{if} \quad L > 1.0, \\ 2\cos^2(\pi/2L) & \text{if} \quad L \leq 1.0 \end{cases}$$

  where

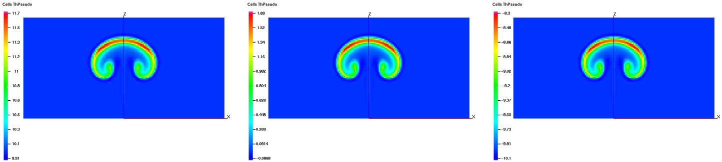
$$L = \sqrt{\frac{(x - x_c)^2}{x_r^2} + \frac{(z - z_c)^2}{z_r^2}}$$

  with $x_c = 10000\text{m}$, $z_c = 2000\text{m}$, and $x_r = z_r = 2000\text{m}$
- Simulation time 1000 s

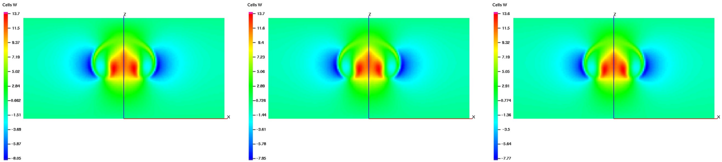
- Perturbation potential temperature for the anelastic case



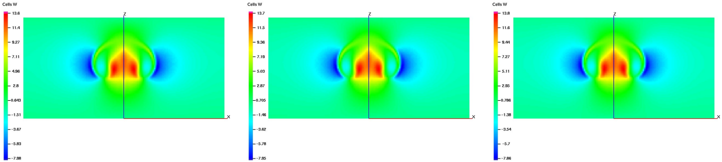

- Perturbation potential temperature for the pseudo incompressible case



- Vertical velocity for the anelastic case



- Vertical velocity for the pseudo incompressible case



- Comparison of minimal and maximal vertical velocity

| | | |
|---|---|---|
| Compressible | | |
| 300 | 13.700 | -7.922 |
| Anelastic | | |
| 290 | 13.725 | -8.046 |
| 300 | 13.733 | -7.946 |
| 310 | 13.596 | -7.774 |
| Pseudo | | |
| 290 | 13.580 | -7.983 |
| 300 | 13.684 | -7.948 |
| 310 | 13.761 | -7.864 |

- Domain 400000 m by 30.000 m
- $\Delta x = \Delta z = 200$m, $\Delta t = 5$s
- Initial conditions for the unperturbed environment (calm, hydrostatic, nonisothermal)
- Until 11000 m a lapse rate of 6.5Km$^{-1}$, above isothermal
- Perturbation

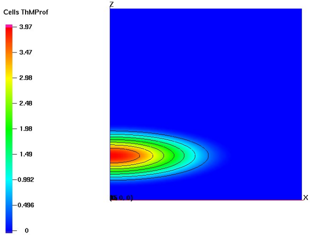
$$\Delta\theta = \begin{cases} 0 & \text{if} \quad L > 1.0, \\ 4\cos^2(\pi/2L) & \text{if} \quad L \leq 1.0 \end{cases}$$

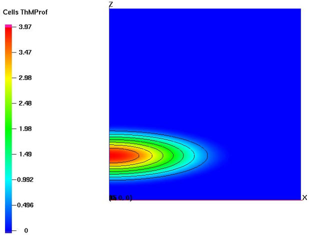
  where

$$L = \sqrt{\frac{(x - x_c)^2}{x_r^2} + \frac{(z - z_c)^2}{z_r^2}}$$

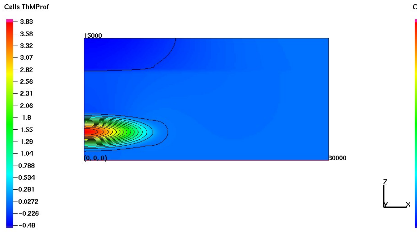
  with $x_c = 200000$m, $z_c = 3500$m, $x_r = 10000$m, and $z_r = 2500$m
- Simulation time 1000 s

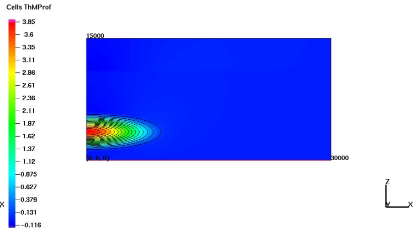
- Initial perturbations of potential temperature compressible left and anelastic right



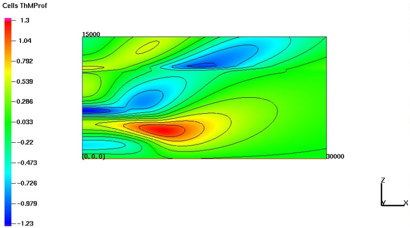



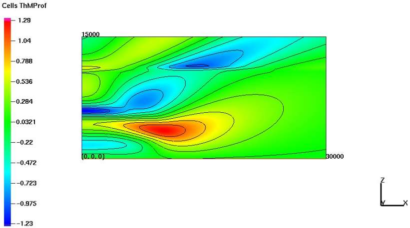
- Potential temperature perturbation after 1 min compressible left and anelastic right



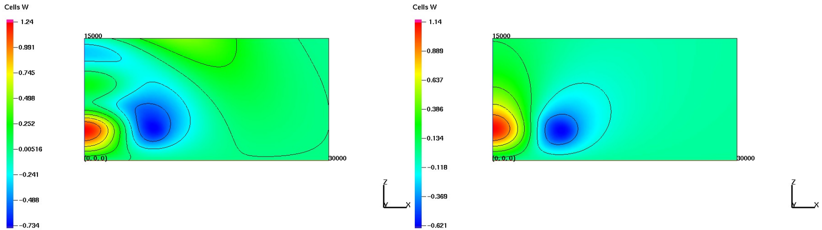
- Potential temperature perturbation after 10 min compressible left and anelastic right



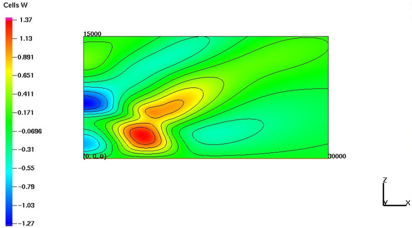



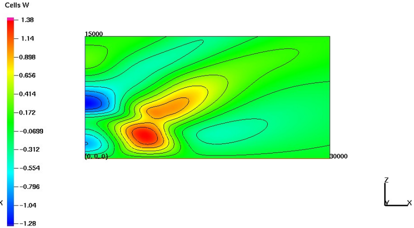
## Vertical velocity

- Vertical velocityafter 1 min compressible left and anelastic right



- Vertical velocity after 10 min compressible left and anelastic right



Need we anelastic approximations

- How to choose the background profiles
- Sound waves can be eliminated by suitable time integration schemes
- Same numerical schemes for sound proofed approximations and full equations
- The linear solver for the for the sound part of the Jacobian is the crucial bottleneck
- Pseudo incompressible numerics are unstable for larger time steps

- Looking for more optimal methods
- Partial linearly implicit Peer and Rosenbrock methods
- Stability analysis for approximated Jacobians
- Dispersion analysis of the numerical schemes for linearized problems
- Adapted pressure solvers for different grid configurations