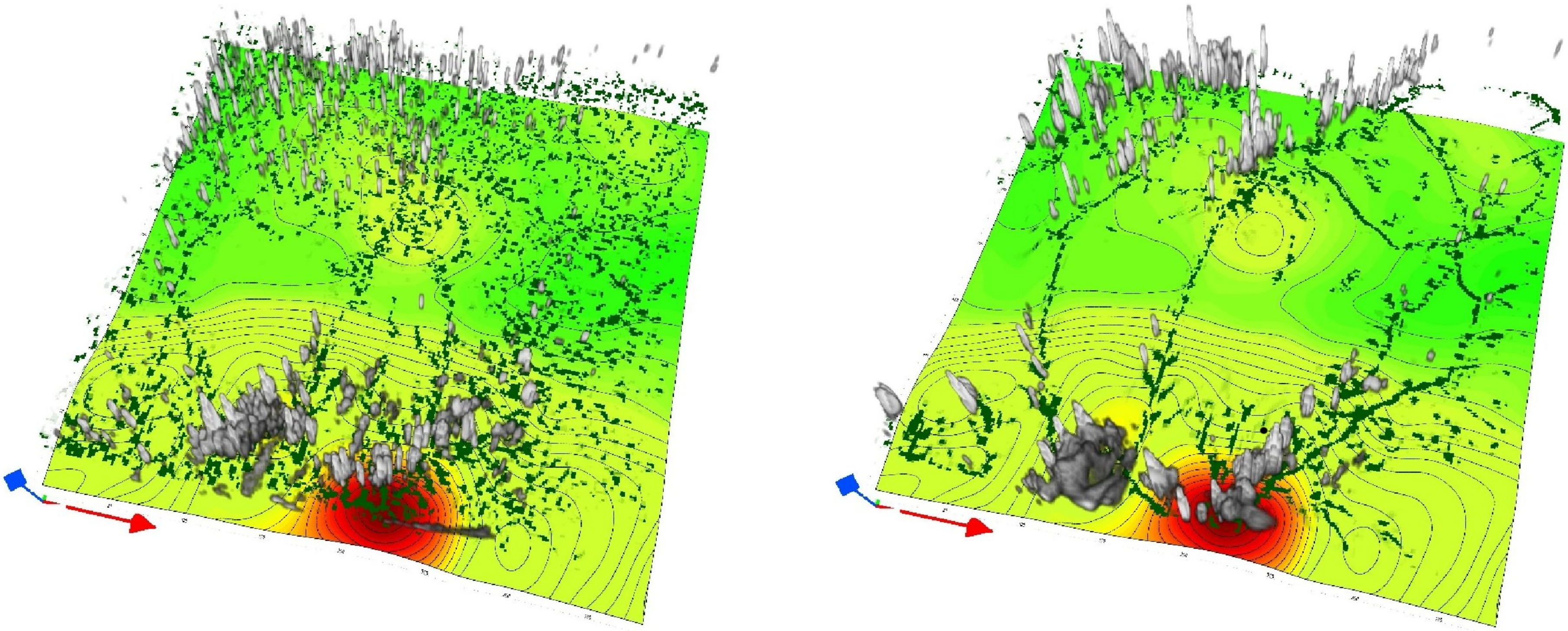
Zbigniew P. Piotrowski *,**
Piotr K. Smolarkiewicz *

Rayleigh-Benard convection – effects of Prandtl number anisotropy

*National Center for Atmospheric Research, Boulder, Colorado, U.S.A.
** Currently NCAR Geophysical Turbulence Program postdoc. On the leave from Institute for Meteorology and Water Management, Warsaw, Poland

NCAR is sponsored by the National Science Foundation

Motivation



Structure of simulated convection over heated realistic terrain.

Vertical velocities after 6h of simulated time are shown within the PBL depth. Grey iso-surfaces represent clouds, and dark green patterns mark updrafts at boundary layer top. Isolines and other colors show the topography. ***The only difference between the two simulations is the effective viscosity of numerical advection.***

Rayleigh number :

$$Ra = \frac{g\Delta\overline{\theta}h^3}{\overline{\theta}\nu\nu_\theta}$$

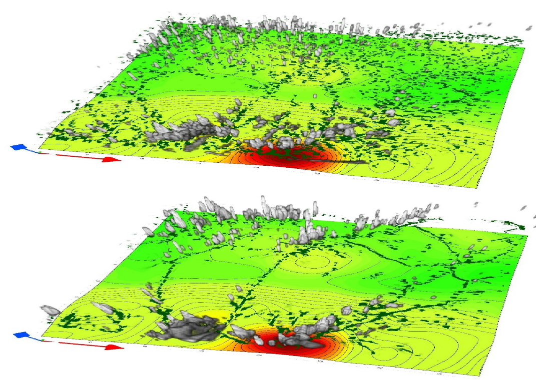
g – gravity acceleration
h – fluid layer thickness
ν – kinematic viscosity
ν_θ – thermal diffusivity
$\Delta\theta / \theta$ – pot. temperature, relative change over h

¿

Ra measures relative magnitude of buoyancy and viscous forces

rigid/stress-free
lower/upper

Ra_c=1100.657



>> critical

≈ critical

In the dry atmosphere:

h= 1000 m
ν = 1.7 x 10^{-5} m^2/s
ν_θ =1.9 x 10^{-5} m^2/s
$\Delta\theta / \theta = O(10^{-3})$

→ **Ra ≈ O(10^{16})**

Thus, how to explain cellular convection ?

Modified definition (Jeffreys, 1928, Priestley 1962, Ray 1965, Sheu

$$Ra = \frac{g \Delta \overline{\theta} h^3}{\overline{\theta} K_m^2}$$

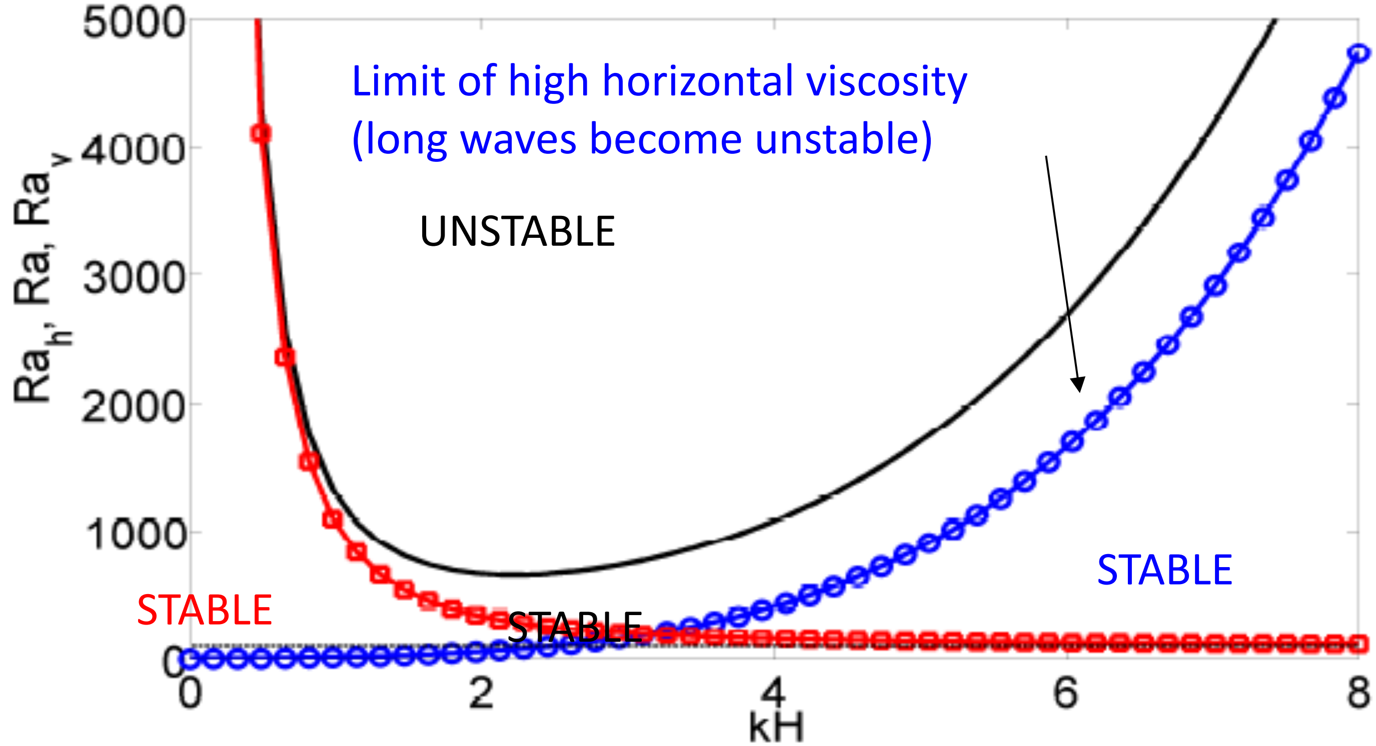
Km can be different in the horizontal and in the vertical.

Possible sources of K_m and *Ra* anisotropy in virtual reality

- Explicit anisotropic filtering
- Using numerical schemes with different dissipative properties in the horizontal and in the vertical
- Numerical dissipation ~**V** (flow magnitude), as oppose to ~∂**V**; e.g., first-order upwinding, or composite schemes
- Prandtl number anisotropy – resulting from, for example, disparate approximations to governing equations

Linear theory – effect of viscosity anisotropy

Piotrowski et al, "On numerical realizability of thermal convection",Vol. 228, 2009



Asymptotic marginal stability relations for a finite Prandtl number and $v_h = v_v$ (black solid), $v_v = 0$ (blue circles) and $v_h = 0$ (red squares). Respective Rayleigh numbers Ra_h, Ra and Ra_v are shown in function of the horizontal wave number. Stability region is below the curves.

Generalized governing equations for Rayleigh-Benard convection for anisotropic viscosity AND Prandtl number anisotropy

Hadamard (entrywise) product

Momentum eq. $$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\phi + g\alpha\theta\nabla z + \mathbf{\Delta} \circ \mathbf{u} \ ,$$

Temperature eq. $$\frac{\partial \theta}{\partial t} = \beta w + \kappa_h \partial_h^2 \theta + \kappa_v \partial_z^2 \theta \ ,$$

Continuity eq. $$\nabla \cdot \mathbf{u} = 0 \ ,$$

Vector laplacian $$\mathbf{\Delta} := (\widehat{\nu}_h \partial^2 + \triangle_0,\ \widehat{\nu}_h \partial^2 + \triangle_0,\ \widehat{\nu}_v \partial^2 + \triangle_0)$$

Scalar laplacian $$\triangle_0 := \nu_h \partial_h^2 + \nu_v \partial_z^2 \ , \quad \partial_h^2 := \partial_x^2 + \partial_y^2 \ ,$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\phi + g\alpha\theta\nabla z + \boldsymbol{\Delta} \circ \mathbf{u} \;,$$
$$\frac{\partial \theta}{\partial t} = \beta w + \kappa_h \partial_h^2 \theta + \kappa_v \partial_z^2 \theta \;,$$
$$\nabla \cdot \mathbf{u} = 0 \;,$$

Disparate approximations to momentum equations (full set of stress tensor entries)

$$\boldsymbol{\Delta} := (\widehat{\nu}_h \partial^2 + \Delta_0, \widehat{\nu}_h \partial^2 + \Delta_0, \widehat{\nu}_v \partial^2 + \Delta_0)$$
$$\Delta_0 := \nu_h \partial_h^2 + \nu_v \partial_z^2 \;, \quad \partial_h^2 := \partial_x^2 + \partial_y^2 \;,$$

Anisotropic viscosity (coefficients at diagonal entries of stress tensor)

Analogy - Equation set for R-B convection in nematic liquid crystals:

STABILITY OF NEMATIC LIQUID CRYSTALS UNDER A TEMPERATURE GRADIENT. CALCULATIONS FOR PAA†

ATTILA AŞKAR‡
The Scientific and Technical Research Council T.B.T.A.K., Inşa
Turkey

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\phi + g\alpha\theta\nabla z + \mathbf{\Delta}\circ\mathbf{u} \ ,$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa_h \partial_h^2\theta + \kappa_v \partial_z^2\theta \ ,$$

$$\nabla\cdot\mathbf{u} = 0 \ ,$$

$$v_{1,1} + v_{3,3} = 0$$

$$t_{11,1} + t_{31,3} + \rho f_1 - \rho\left(\frac{\partial v_1}{\partial t} + v_1 v_{1,1} + v_3 v_{1,3}\right) = 0$$

$$t_{13,1} + t_{33,3} + \rho f_3 - \rho\left(\frac{\partial v_3}{\partial t} + v_1 v_{3,1} + v_3 v_{3,3}\right) = 0$$

$$m_{12,1} + m_{32,3} - (t_{13} - t_{31}) + \rho l_2 = 0$$

$$\frac{\partial T}{\partial t} + v_1 T_{,1} + v_3 T_{,3} + q_{1,1} + q_{3,3} = 0.$$

Similarly the constitutive relations read

$$t_{11} = -p + (a_{1111} - a_{1133}) v_{1,1}$$
$$t_{33} = -p + (a_{3333} - a_{3311}) v_{3,3}$$
$$t_{13} = a_{1331} v_{1,3} + a_{1313} v_{3,1} + (a_{1313} - a_{1331})\dot{\psi}_2$$
$$t_{31} = a_{3131} v_{1,3} + a_{3113} v_{3,1} + (a_{3113} - a_{3131})\dot{\psi}_2$$
$$m_{12} = B_{2121}\psi_{2,1}$$
$$m_{32} = B_{2323}\psi_{2,3}$$
$$q_1 = -(k_{11}T_{,1} + k_{13}T_{,3})$$
$$q_3 = -(k_{31}T_{,1} + k_{33}T_{,3})$$

These equation sets are very similar
in viscous tensor formulation,
when L.C. equations are linearized and
microrotation of crystals neglected.

Applying operator of rotation to momentum equations:

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) = g\alpha \nabla \times \theta \nabla z + \boxed{\triangle_0 \nabla \times \mathbf{u}} + \boxed{\triangle \nabla \times (\widehat{\nu} \circ \mathbf{u})}$$

This term describes possible production of baroclinic vorticity

Taking rotation once again and considering the vertical component:

$$\frac{\partial}{\partial t}\partial^2 w = g\alpha \partial_h^2 \theta + \boxed{\triangle_0 \partial^2 w} + \boxed{(\widehat{\nu}_v \partial_h^2 + \widehat{\nu}_h \partial_z^2)\partial^2 w}$$

Equation set for vertical velocity and potential temperature becomes:

$$\left(\frac{d^2}{dz^2} - k^2\right)\left((\widehat{\nu}_h + \nu_v)\frac{d^2}{dz^2} - (\widehat{\nu}_v + \nu_h)\, k^2 - p\right)\hat{w} = g\alpha k^2 \hat{\theta}$$

$$\left(\kappa_v \frac{d^2}{dz^2} - \kappa_h k^2 - p\right)\hat{\theta} = -\beta \hat{w}\,.$$

Assuming solution in Fourier modes:

$$w = \hat{w}(z)\exp[i(k_x x + k_y y) + pt] \ ,$$
$$\theta = \hat{\theta}(z)\exp[i(k_x x + k_y y) + pt] \ ; \quad k^2 := k_x^2 + k_y^2 \ , \quad i := \sqrt{-1}$$

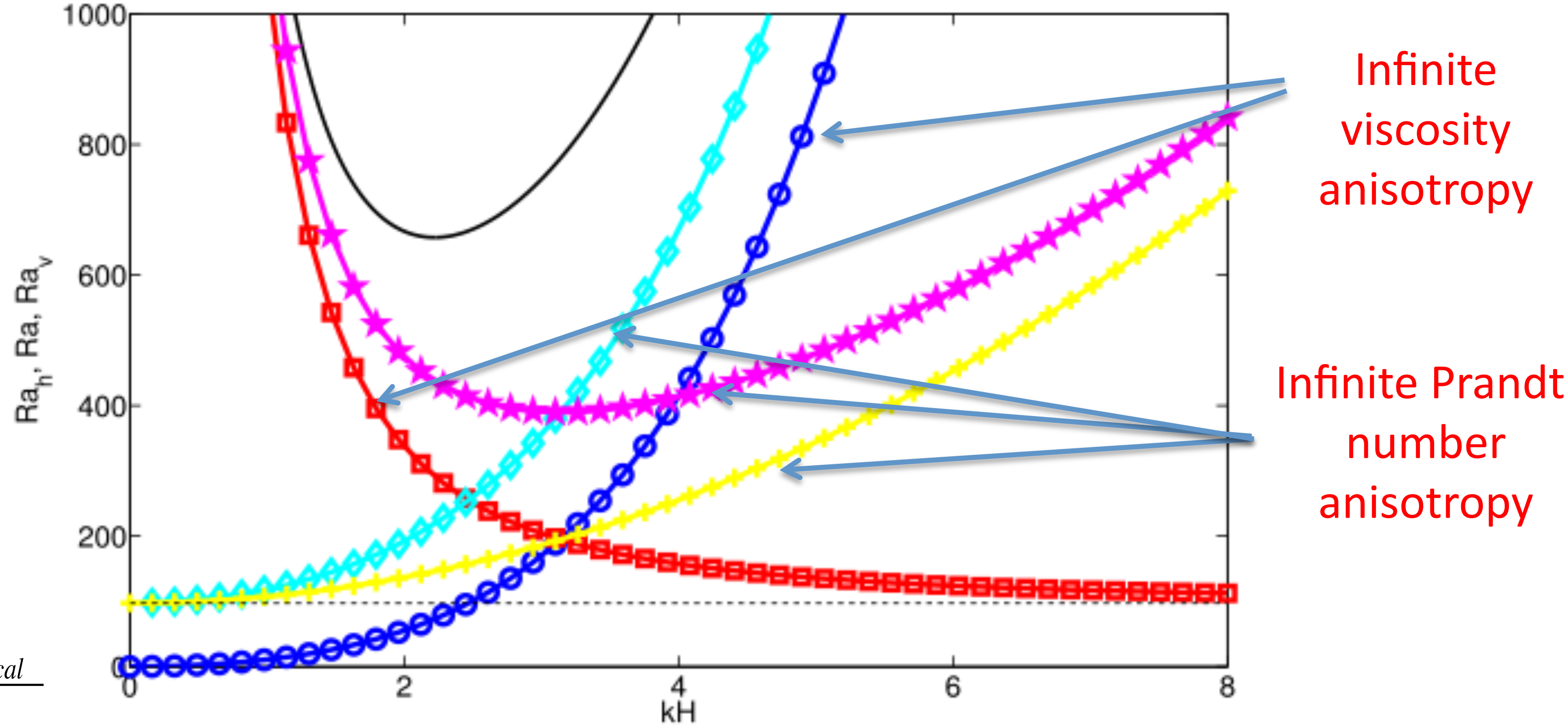
$$\left(\frac{d^2}{dz^2} - k^2\right)\left((\widehat{\nu}_h + \nu_v)\frac{d^2}{dz^2} - (\widehat{\nu}_v + \nu_h)\,k^2 - p\right)\hat{w} = g\alpha k^2 \hat{\theta}$$

$$\left(\kappa_v \frac{d^2}{dz^2} - \kappa_h k^2 - p\right)\hat{\theta} = -\beta \hat{w} \, .$$

Note that number of parameters is now effectively reduced, regardless if we consider $\vec{\nu}$, $\vec{\hat{\nu}}$ or both.

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\nu_{veff}\frac{d^2}{dz^2} - \nu_{heff}k^2 - p\right)\left(\kappa_v \frac{d^2}{dz^2} - \kappa_h k^2 - p\right)\hat{w}$$
$$= -g\alpha k^2 \beta \hat{w} \, .$$

Linear theory
effects of viscosity anisotropy AND Prandtl number anisotropy

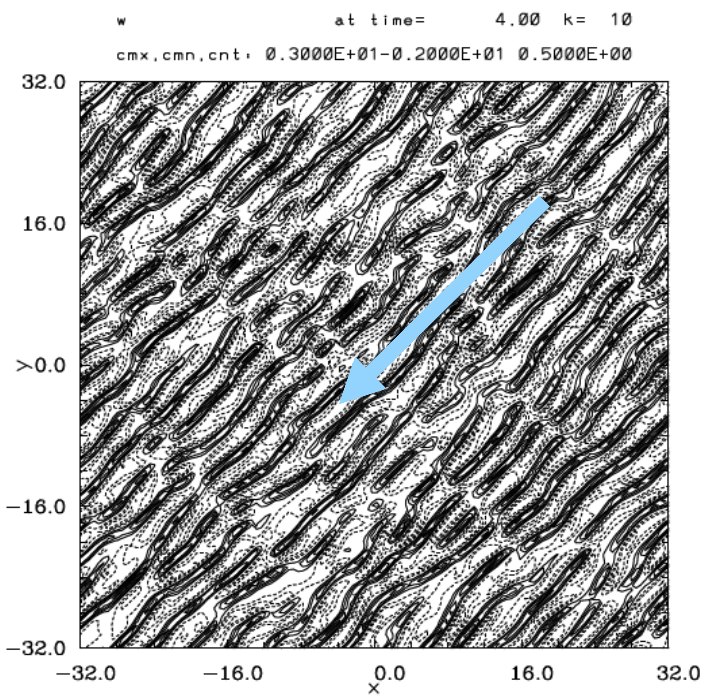


$$r_\upsilon = \frac{\upsilon_{vertical}}{\upsilon_{horizontal}}$$

Fig. 1. Asymptotic marginal stability relations for viscosities $\nu_h = \nu_v$ and thermal diffusivities $\kappa_h = \kappa_v$ (solid), viscosity anisotropy ratios $r_{\nu,\kappa} = 0$ (blue circles), $r_{\nu,\kappa} = \infty$ (red squares), $r_\nu = 0, r_\kappa = 1$ (cyan diamonds), $r_\nu = \infty, r_\kappa = 1$ (magenta stars), $r_\nu = \infty, r_\kappa = 0$ (yellow plus sign); here h and v denote respective values in the horizontal and the vertical. Corresponding Rayleigh numbers Ra are shown as functions of the non-dimensional horizontal wave number kH. For each curve the stability region lies beneath. The square, diamond and plus-sign asymptotics tend to the π^4 limit.

Example of numerical substantiation

Series of LES using the EULAG model



dz=50 m

dx=dy≈500 m

V = [-10,-10] m/s

Heat flux

hfx≈200 W/m^2

Flat lower boundary, doubly periodic horizontal domain, Boussinesq option

Reference setup alludes to contemporary, mesoscale cloud-resolving NWP

Anisotropy viscosity effects

Z.P. Piotrowski et al. / Journal of Computational Physics 228 (2009) 6268–6290

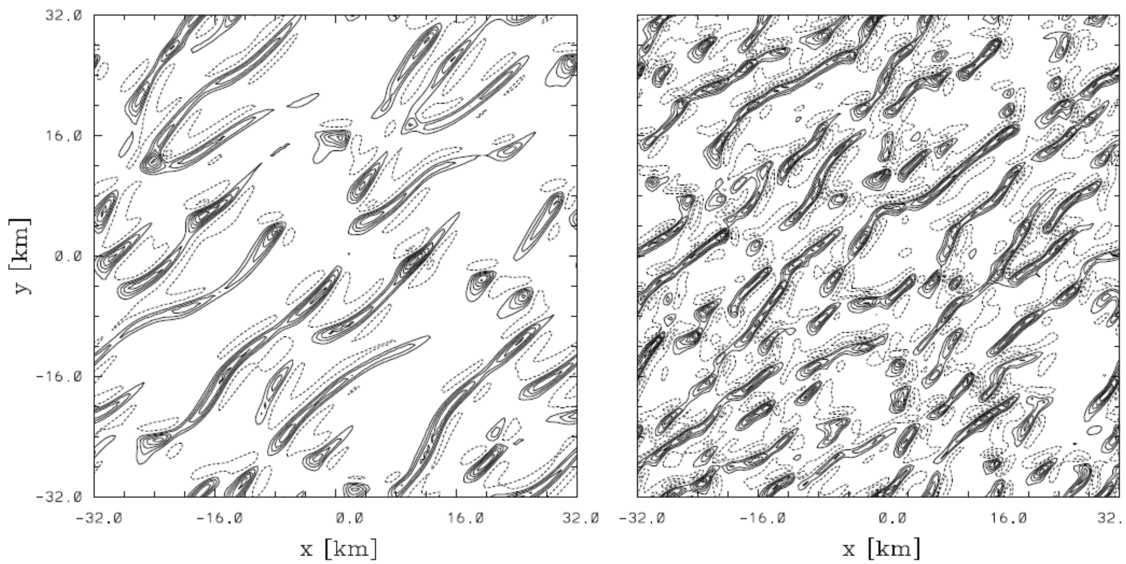
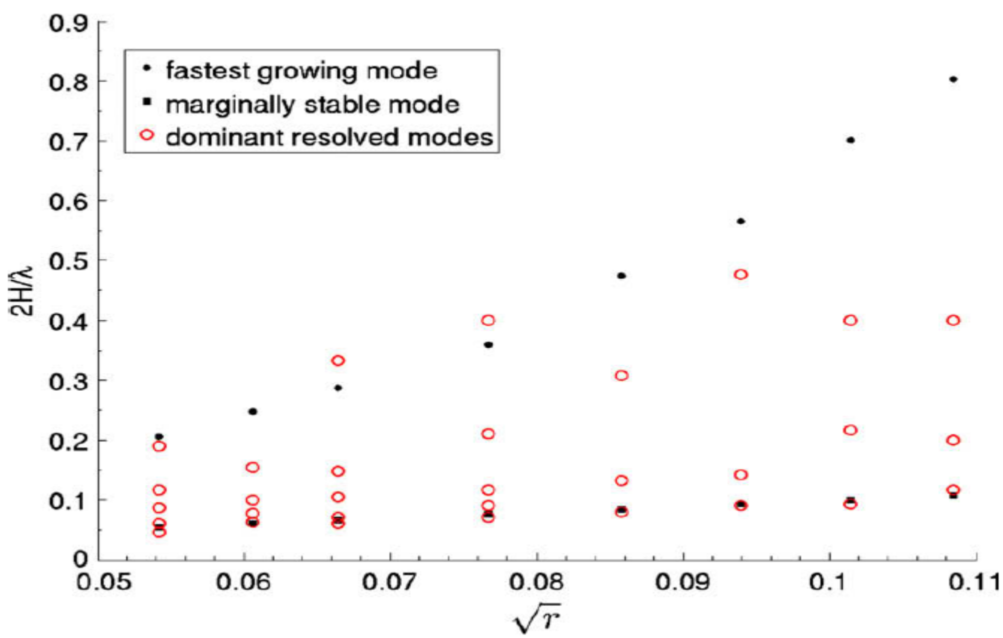
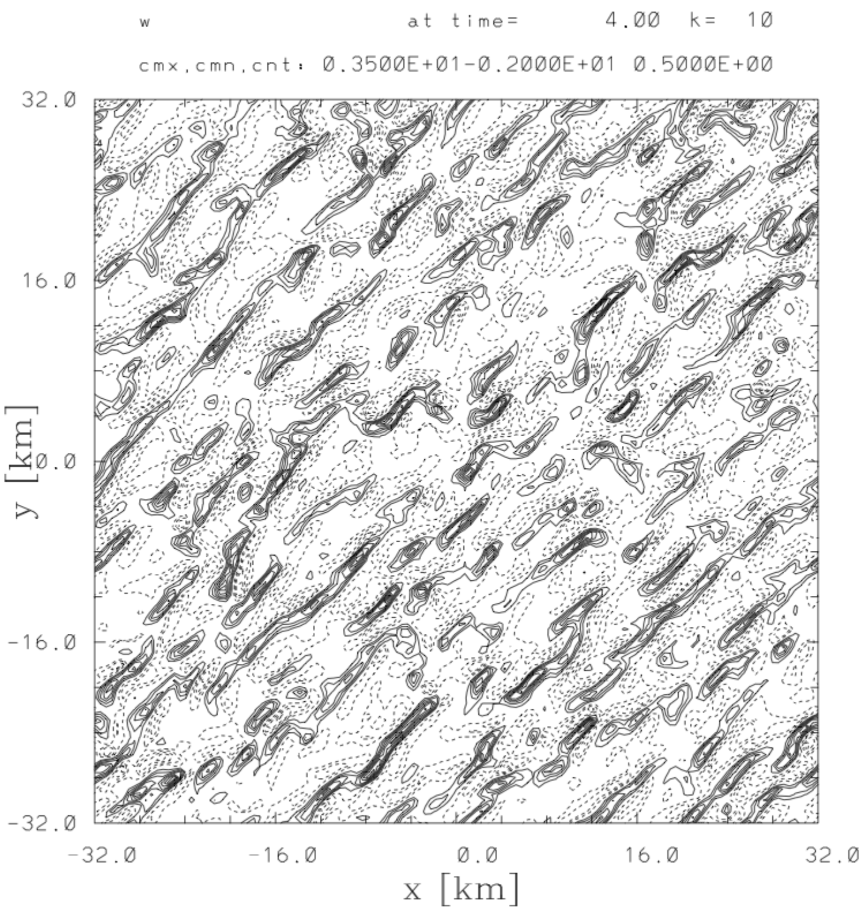


Fig. 20. Vertical velocity as in lower-left panel of Fig. 17 but for $r = 2^l/170$ with $l = -1, 1$, respectively.

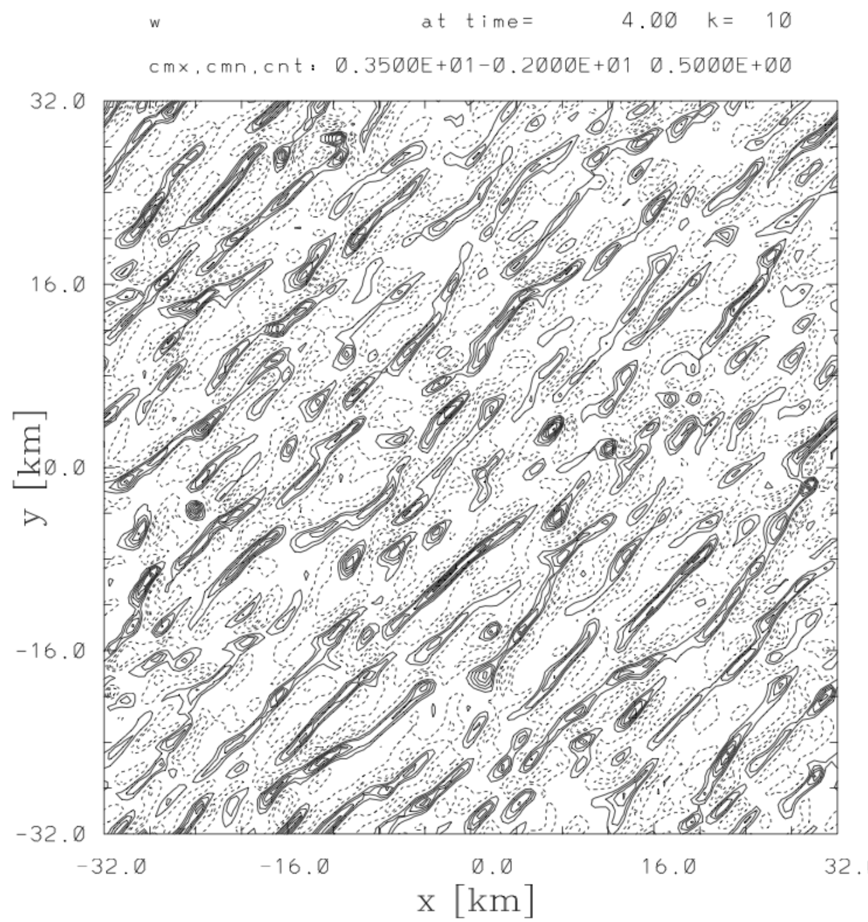


Disparate approximations and anisotropic viscosity effect combined

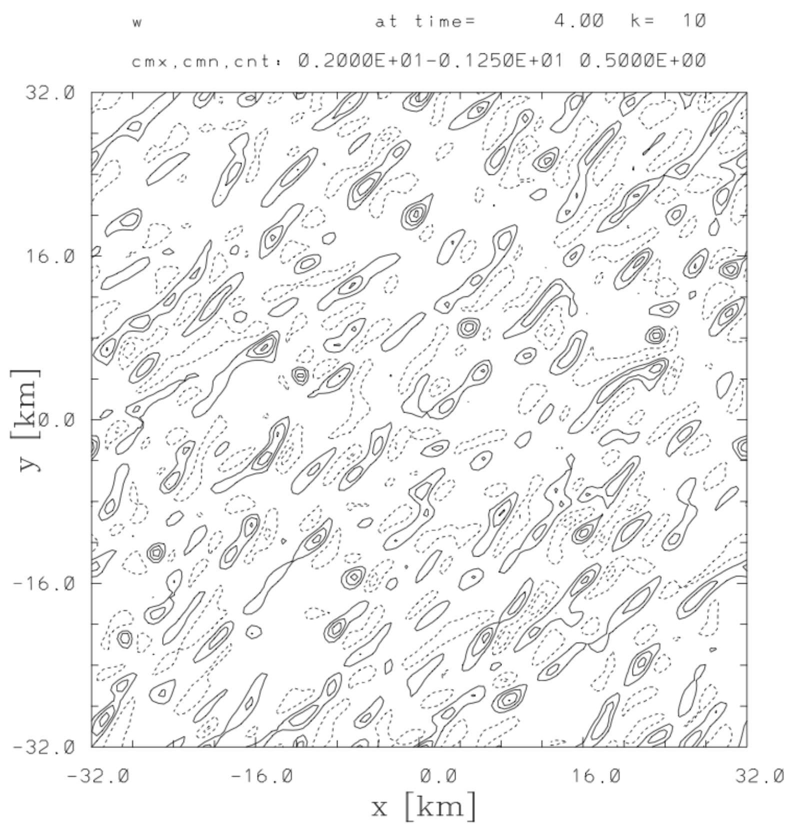
Momentum equations computed with Upwind at every 4th step, MPDATA for temperature equation



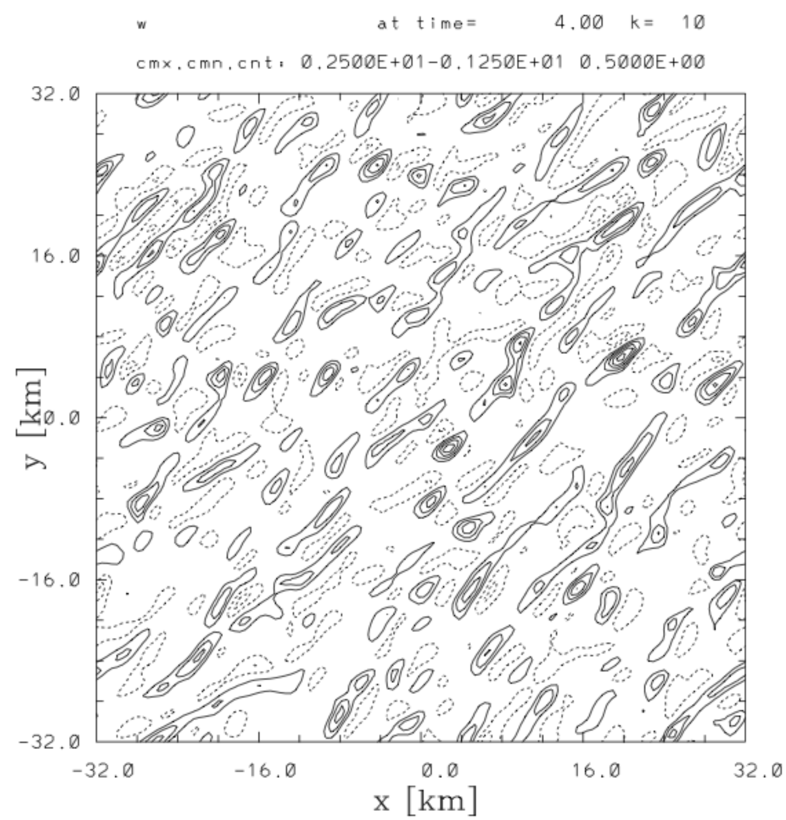
Momentum equations computed with MPDATA, Upwind at every 4th step for temperature equation



3D 1-2-1 filter

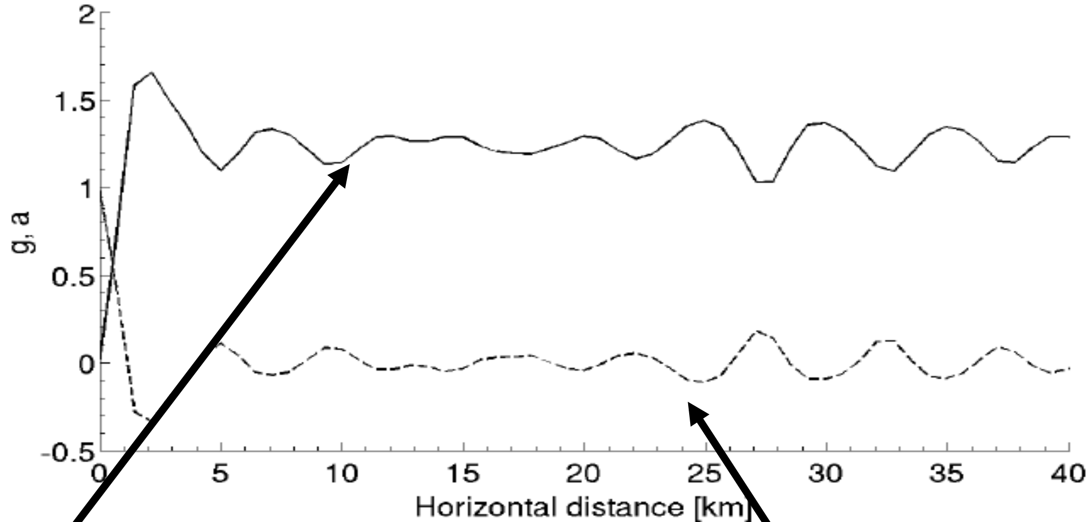


2D spatial 1-2-1 filter



Updrafts are stronger with the 2D spatial filtering of momentum equations. Is the convection more anisotropic as well ?

Convective structure quantification



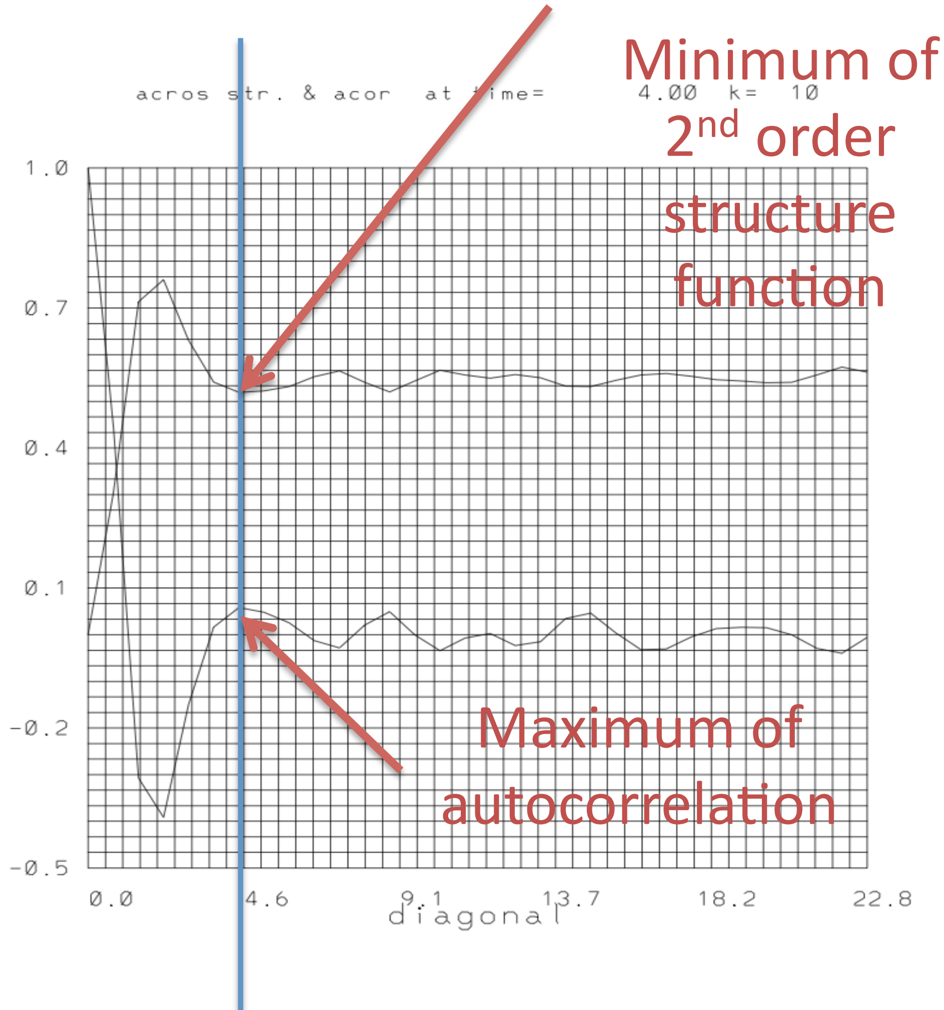
Second order structure function

$$g(\delta x, \delta y) = \frac{1}{nx \cdot ny} \Sigma_{i=1}^{nx} \Sigma_{j=1}^{ny} [w(x_i + \delta x, y_j + \delta y) - w(x_i, y_j)]^2$$

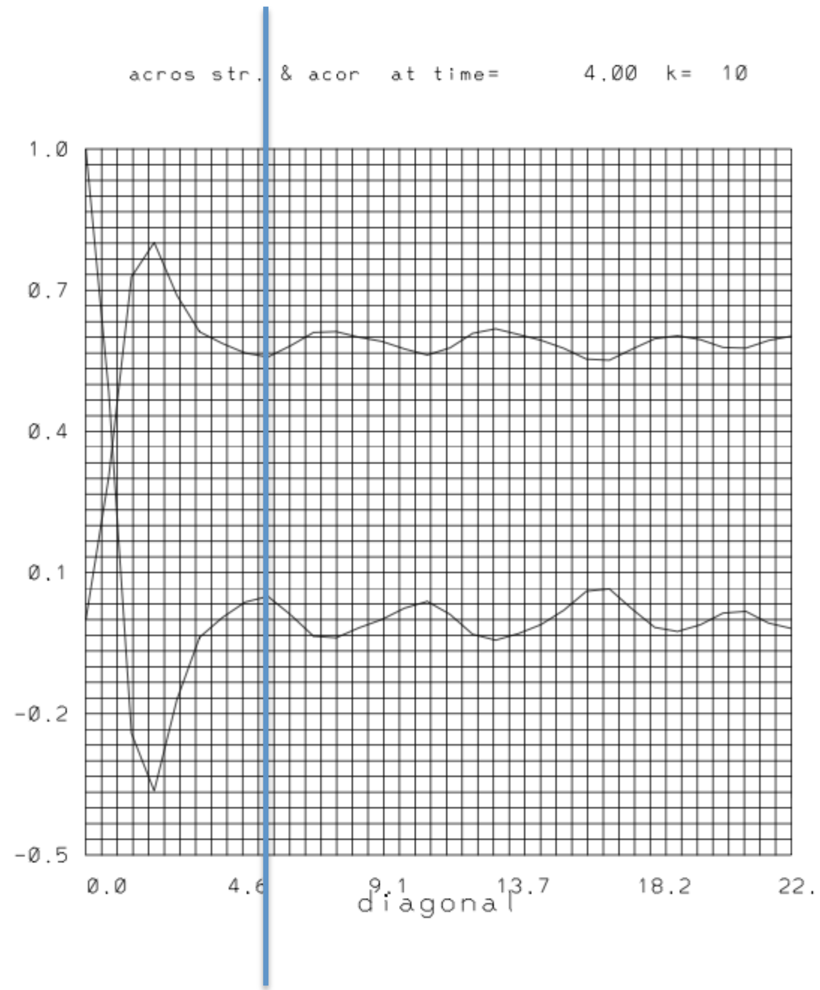
Autocorrelation function

$$a(\delta x, \delta y) = \frac{1}{\sigma^2} \Sigma_{i=1}^{nx} \Sigma_{j=1}^{ny} [w(x_i + \delta x, y_j + \delta y) - \overline{w}][w(x_i, y_j) - \overline{w}]$$

Diagonal structure function and autocorrelation



3D filtering



2D spatial filtering

2D spatial filtering produces larger structures than full 3D filtering !

Conclusions

- Cellular convection simulated with meso- and large-scale models may be only a spurious result of the effective anisotropic viscosity
- The linear theory has a skill to quantify the anisotropic viscosity effects
- Implicit numerical viscosity and dispersion are well known. There appears to be a need for appreciating ``implicit numerical topology'' while analyzing under-resolved convective structures and cloud coverage

Conclusions

- Prandtl number anisotropy, similar to anisotropic viscosity, can modify marginal stability of realized R-B convection
- Linear theory successfully describes joint influence of anisotropic entries in viscous stress tensor and the Prandtl number anisotropy, reducing the number of parameters
- Prandtl number anisotropy can be realized with controlled disparate approximations to the governing equations