Numerical merits of anelastic models

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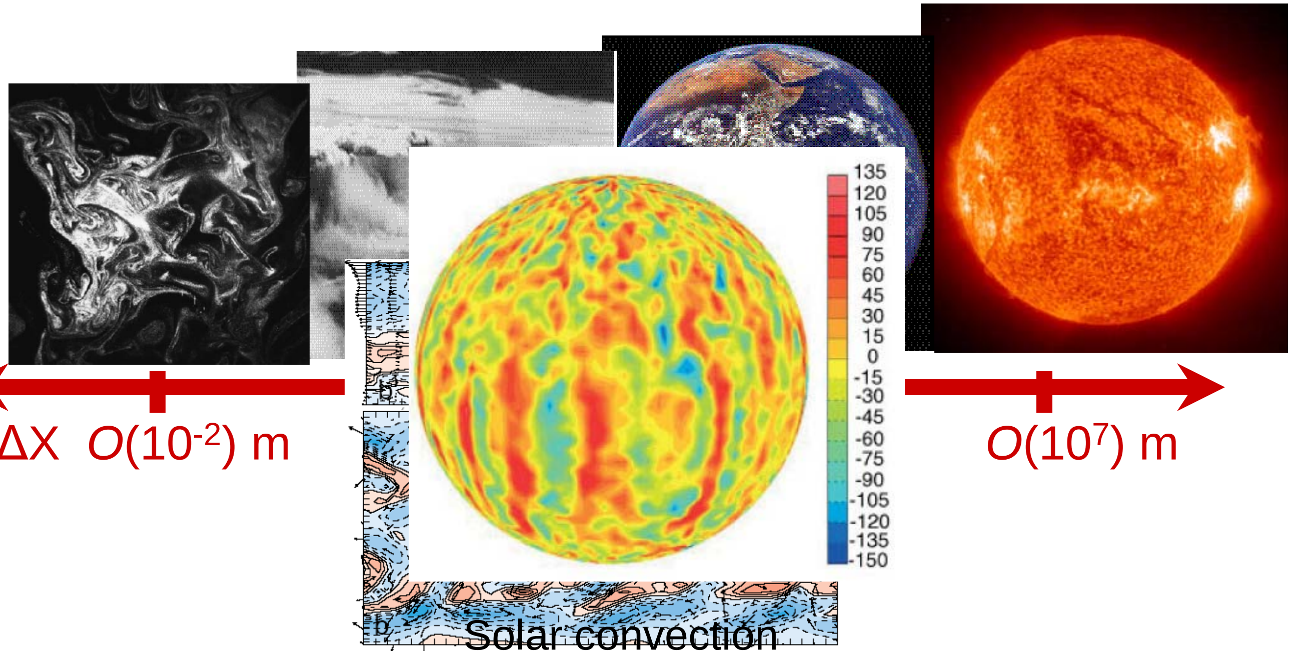

ΔX $O(10^{-2})$ m

$O(10^{7})$ m

Solar convection

Cloud turbulence

Global flows

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EULAG's key options

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Supported (*operational in demo codes*) and "private" (*hidden or available in some clones*)

Two options for integrating fluid PDEs with nonoscillatory forward-in-time Eulerian (control-volume wise) & semi Lagrangian (trajectory wise) model algorithms, but also Adams-Bashforth Eulerian scheme for basic dynamics

Preconditioned non-symmetric Krylov-subspace elliptic solver, but also the pre-existing MUDPACK experience. Notably, for simple problems in Cartesian geometry, the elliptic solver is direct (viz. spectral preconditioner).

Generalized time-dependent curvilinear coordinates for grid adaptivity to flow features and/or complex boundaries,but also the immersed-boundary method.

PDEs of fluid dynamics (*comments on nonhydrostacy ~ g, xy 2D incompressible Euler*):

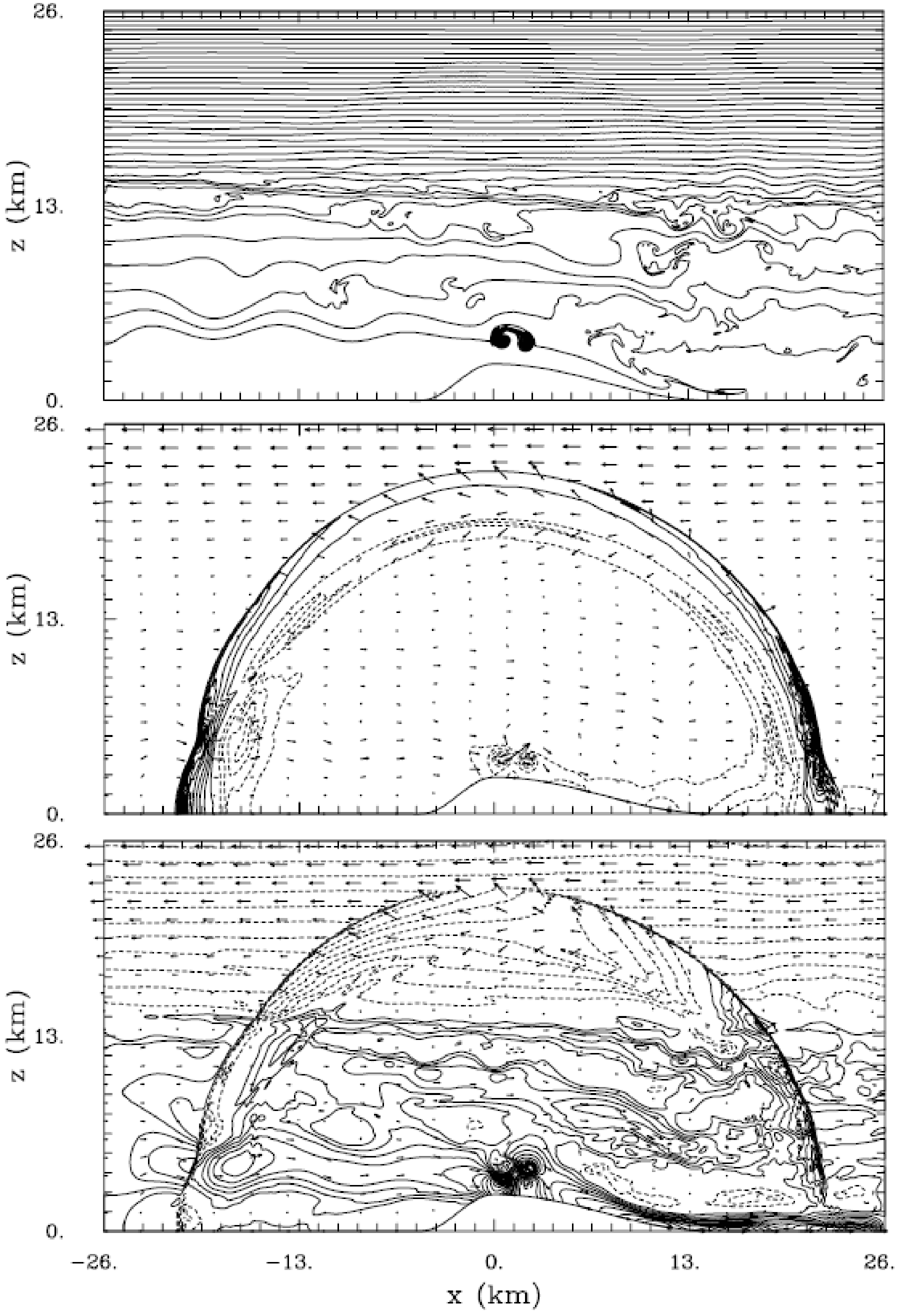
- Anelastic (incompressible Boussinesq, Ogura-Phillips, Lipps-Hemler, Bacmeister-Schoeberl, Durran)
- Compressible Boussinesq
- Incompressible Euler/Navier-Stokes' (somewhat tricky)
- Fully compressible Euler/Navier-Stokes' for high-speed flows (3 different formulations).
- Boussinesq ocean model, anelastic solar MHD model, a viscoelastic ``brain'' model, porous media model, sand dunes, dust storma, etc.

Physical packages:

- Moisture and precipitation (several options) and radiation
- Surface boundary layer
- ILES, LES, LES, DNS

Analysis packages: momentum, energy, vorticitiy, turbulence and moisture budgets

Example of anelastic and compressible PDEs



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Numerics related to the "geometry" of an archetype fluid PDE/ODE

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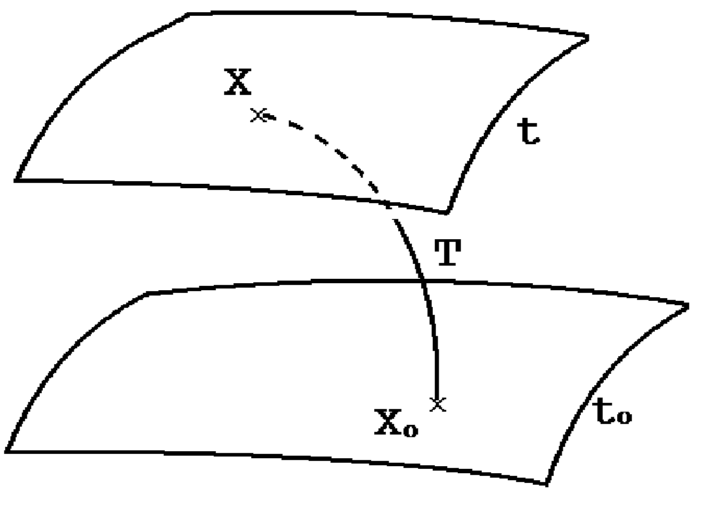
Eulerian conservation law   *Lagrangian* evolution equation

$$\frac{\partial \rho^* \psi}{\partial \overline{t}} + \overline{\nabla} \bullet (\rho^* \overline{\mathbf{v}}^* \psi) = \rho^* R \quad \Leftrightarrow \quad \frac{d\psi}{d\overline{t}} = R$$

$\psi \equiv v^j \text{ or } \theta'$ Kinematic or thermodynamic variables, R the associated rhs

Either form is approximated to the second-order using a template algorithm:

$$\boxed{\psi_{\mathbf{i}}^{n+1} = LE_{\mathbf{i}}(\psi^n + 0.5\Delta t R^n) + 0.5\Delta t R_{\mathbf{i}}^{n+1}}$$



Temporal differencing for either anelastic or compressible PDEs, depending on definitions of *G* and *Ψ*

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$$\boxed{\frac{\partial G\Psi}{\partial t} + \nabla \cdot (\mathbf{v}\Psi) = GR}$$

Forward in time temporal discretization

$$\frac{G^{n+1}\Psi^{n+1} - G^n\Psi^n}{\delta t} + \nabla \cdot (\mathbf{v}^{n+1/2}\Psi^{n}) = (GR)^{n+1/2}$$

Second order Taylor sum expansion about =nΔt

$$\frac{\partial G\Psi}{\partial t} + \nabla \cdot (\mathbf{v}\Psi) = GR - \nabla \cdot \left[\frac{\delta t}{2}G^{-1}\mathbf{v}(\mathbf{v}\cdot\nabla\Psi) + \frac{\delta t}{2}G^{-1}\left(\frac{\partial G}{\partial t} + \nabla\cdot\mathbf{v}\right)\mathbf{v}\Psi\right] + \nabla \cdot \left(\frac{\delta t}{2}\mathbf{v}R\right) + \mathcal{O}(\delta t^2)$$

Compensating first error term on the rhs is a responsibility of an FT advection scheme (e.g. MPDATA). The second error term depends on the implementation of an FT scheme

$$\boxed{\Psi_i^{n+1} = LE_i(\Psi^n + 0.5\Delta t R^n) + 0.5\Delta t R_i^{n+1}}$$

explicit/implicit rhs

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implicit: all forcings are assumed to be unknown at *n*+1

$$\psi_{\mathbf{i}}^{n+1} = LE_{\mathbf{i}}(\psi^n + 0.5\Delta t R^n) + 0.5\Delta t R_{\mathbf{i}}^{n+1}$$

$\Rightarrow$ system implicit with respect to all dependent variables.

On grids co-located with respect to all prognostic variables, it can be inverted algebraically to produce an elliptic equation for pressure

$$\left\{ \frac{\Delta t}{\rho^*} \overline{\nabla} \cdot \rho^* \widetilde{\mathbf{G}}^T \left[\widehat{\hat{\mathbf{v}}} - (\mathbf{I} - 0.5\Delta t \widehat{\mathbf{R}})^{-1} \widetilde{\mathbf{G}} (\overline{\nabla} \pi'') \right] \right\}_{\mathbf{i}} = 0$$

solenoidal velocity $\overline{\mathbf{v}}^s \equiv \overline{\mathbf{v}}^* - \frac{\partial \overline{\mathbf{x}}}{\partial t}$ *contravariant velocity* $\dot{\overline{\mathbf{v}}}^* \equiv d\overline{\mathbf{x}}/d\overline{t} \equiv \dot{\overline{\mathbf{x}}}$

$$\widetilde{\mathbf{G}}^T \left[\widehat{\hat{\mathbf{v}}} - (\mathbf{I} - 0.5\Delta t \widehat{\mathbf{R}})^{-1} \widetilde{\mathbf{G}} (\overline{\nabla} \pi'') \right] \equiv \overline{\mathbf{v}}^s$$

Boundary conditions on π'' Imposed on $\overline{\mathbf{v}}^s \bullet \mathbf{n}$ subject to the integrability condition

$$\int_{\partial\Omega} \rho^* \overline{\mathbf{v}}^s \bullet \mathbf{n} d\sigma = 0$$

Boundary value problem is solved using nonsymmetric Krylov subspace solver
- a preconditioned generalized conjugate residual GCR(*k*) algorithm
(Smolarkiewicz and Margolin, 1994; Smolarkiewicz et al., 2004)

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$$u^{n+1} = \hat{u} - \frac{\Delta t}{2}\frac{\partial \phi}{\partial x} - \frac{\Delta t}{2} F_{n+1}$$

$$w^{n+1} = \hat{w} - \frac{\Delta t}{2}\frac{\partial \phi}{\partial z} + g\frac{\theta'}{\bar{\theta}}\frac{\Delta t}{2}$$

$$\theta^{n+1} = \hat{\theta} - \frac{\Delta t}{2} w^{n+1} S\theta_e$$

$$w^{n+1} = \hat{w} - \left[\frac{\Delta t}{2}\frac{\partial \phi}{\partial z} + \frac{\Delta t}{2}\frac{g}{\bar{\theta}}\left(\hat{\theta} - \frac{\Delta t}{2} w^{n+1} S\theta_e\right)\right]$$

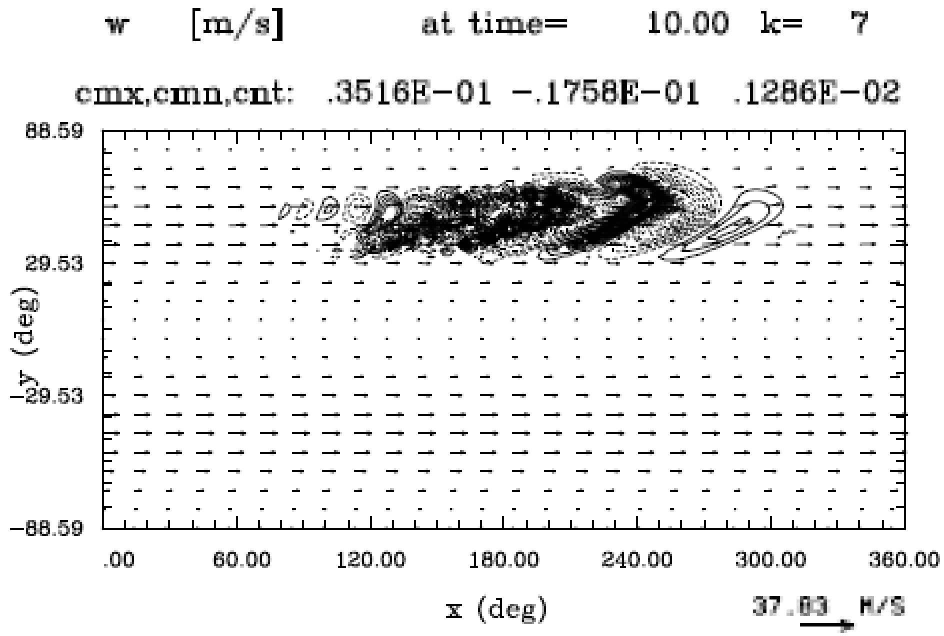
$$w^{n+1}\left(1 + \frac{\Delta t^2}{4}\frac{S\theta_e g}{\bar{\theta}}\right) = \hat{w} + \frac{\Delta t}{2}\frac{g}{\bar{\theta}}\hat{\theta} - \frac{\Delta t}{2}\frac{\partial \phi}{\partial z}$$

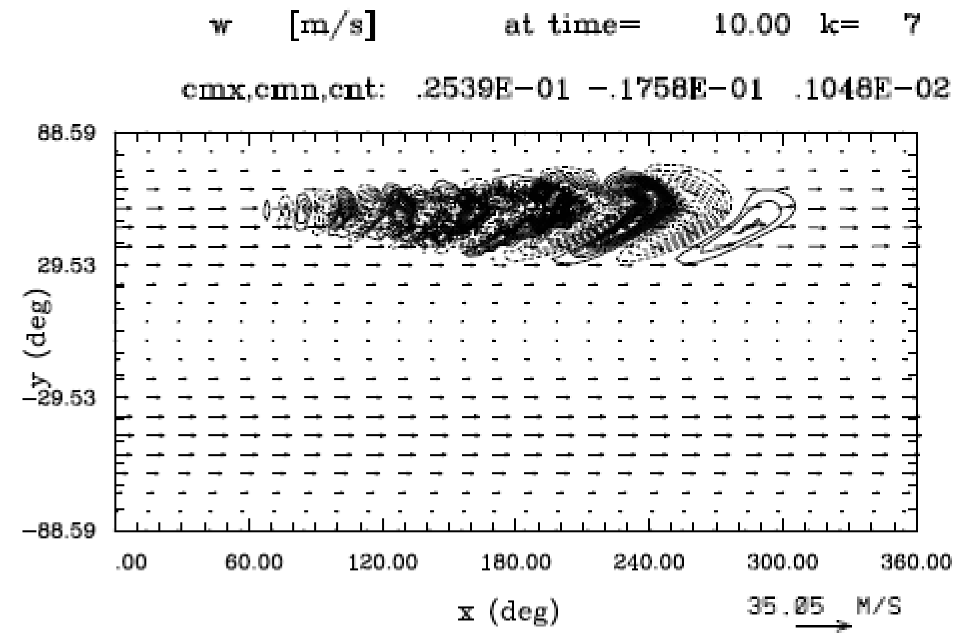
$$w^{n+1} = \frac{\hat{w} + \frac{\Delta t}{2}\frac{g}{\bar{\theta}}\hat{\theta}}{1 + \frac{\Delta t^2}{4}\frac{S\theta_e g}{\bar{\theta}}} - \frac{\Delta t/2}{1 + \frac{\Delta t^2}{4}\frac{S\theta_e g}{\bar{\theta}}}\frac{\partial \phi}{\partial z}$$

$$w^{n+1} = \hat{w} - \frac{\Delta t}{2} F_3 \qquad F_3 = \left(\hat{w} - w^{n+1}\right)\frac{2}{\Delta t}$$

$$F_3 = \left(w^{n+1} - \hat{w}\right)\frac{2}{\Delta t} \qquad = -\left(w^{n+1} - \hat{w}\right)$$

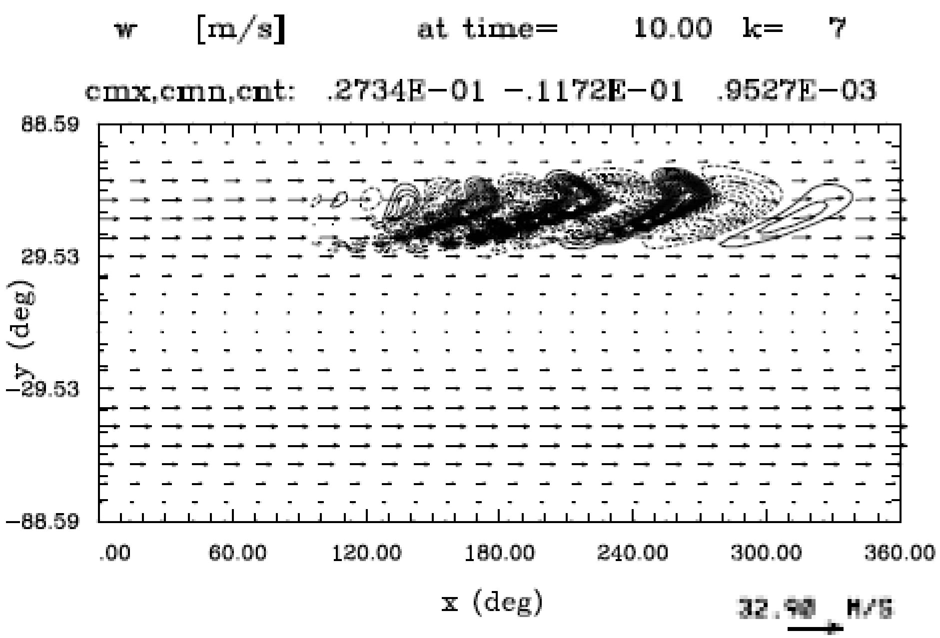
$$F_4 = \frac{\Delta t}{2} w^{n+1} \cdot S \cdot \text{the}.$$

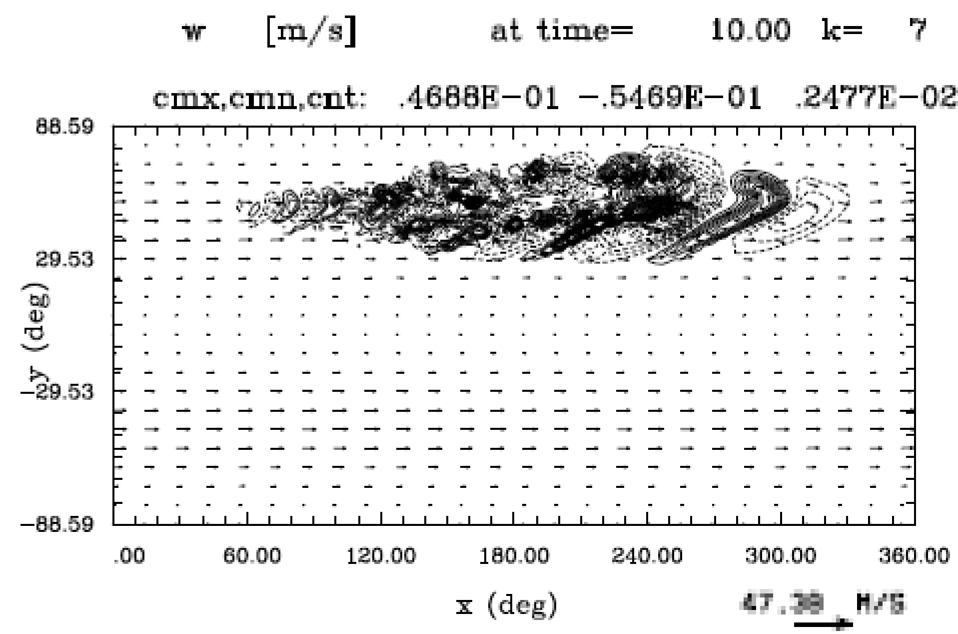




implicit, ~7.5 minutes of CPU

explicit, ~150 minutes of CPU





Implicit/explicit blending, e.g., MHD:

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$$\psi_{\mathbf{i}}^{n+1} = \mathcal{A}_{\mathbf{i}}(\psi^n + 0.5\delta t R^n) + 0.5\delta t R_{\mathbf{i}}^{n+1} \equiv \hat{\psi}_{\mathbf{i}} + 0.5\delta t R_{\mathbf{i}}^{n+1}; \tag{9}$$

- (9) is implicit for all dependent variables in (6)-(8). To retain this proven structure for the MHD system, (9) is executed in the spirit of

$$\Psi_{\mathrm{i}}^{n+1,\nu} = \widehat{\Psi}_{\mathrm{i}} + 0.5\delta t\,\mathbf{L}\Psi\big|_{\mathrm{i}}^{n+1,\nu} + 0.5\delta t\,\mathbf{N}\Psi\big|_{\mathrm{i}}^{n+1,\nu-1} - \nabla\Phi\big|_{\mathrm{i}}^{n+1,\nu} \implies \tag{10}$$

$$\Psi_{\mathrm{i}}^{n+1,\nu} = [\mathbf{I} - 0.5\delta t\,\mathbf{L}]^{-1}\left(\widehat{\Psi} + 0.5\delta t\,\mathbf{N}\Psi\big|^{n+1,\nu-1} - \nabla\Phi^{n+1,\nu}\right)\Big|_{\mathrm{i}} \tag{11}$$

where $\mathbf{L}$ and $\mathbf{N}$ denote linar and nonlinear part of the rhs $\mathbf{R}$, $\Psi \equiv (\mathbf{v},\ \theta',\ \mathbf{B})$, $\Phi \equiv 0.5\delta t(\phi,\ \phi,\ \phi,\ 0,\ \phi^*,\ \phi^*,\ \phi^*)$, and $\nu = 1, .., m$ numbers the iterations.

$$\frac{D\mathbf{v}}{Dt} = -\nabla\pi - \mathbf{g}\frac{\theta'}{\theta_o} + 2\mathbf{v}' \times \Omega + \frac{1}{\mu\rho_o}(\mathbf{B}\cdot\nabla)\mathbf{B} + \mathcal{D}_{\mathbf{v}},$$

$$\frac{D\theta'}{Dt} = -\mathbf{v}\cdot\nabla\theta_e + \mathcal{H} - \alpha\theta',$$

$$\frac{D\mathbf{B}}{Dt} = -\nabla\pi^* + (\mathbf{B}\cdot\nabla)\mathbf{v} - \mathbf{B}(\nabla\cdot\mathbf{v}) + \mathcal{D}_{\mathbf{B}},$$

$$\nabla\cdot(\rho_o\mathbf{v}) = 0,\ \nabla\cdot\mathbf{B} = 0,$$

$$\mathbf{B}_{\rm i}^{n+1,\nu-1/2} = \widehat{\mathbf{B}}_{\rm i} + 0.5\delta t \left[(\mathbf{B}^{n+1,\nu-1/2} \cdot \nabla)\mathbf{v}^{n+1,\nu-1} - \mathbf{B}^{n+1,\nu-1/2} (\nabla \cdot \mathbf{v}^{n+1,\nu-1}) \right]_{\rm i} \; ;$$

$$\begin{aligned}
\theta'|_{\rm i}^{n+1,\nu} &= \widehat{\theta'}_{\rm i} - 0.5\delta t \left(\mathbf{v}^{n+1,\nu} \cdot \nabla\theta_e \right)_{\rm i} \; , \\
\mathbf{v}_{\rm i}^{n+1,\nu} &= \widehat{\mathbf{v}}_{\rm i} + \frac{0.5\delta t}{\mu\rho_s} (\mathbf{B} \cdot \nabla \mathbf{B})_{\rm i}^{n+1,\nu-1/2} \\
&\quad -0.5\delta t \left[\nabla\phi|^{n+1,\nu} + \mathbf{g}\frac{\theta'|^{n+1,\nu}}{\theta_s} + \mathbf{f} \times (\mathbf{v}^{n+1,\nu} - \mathbf{v}_e) \right]_{\rm i} \; , \\
\nabla \cdot (\rho_s \mathbf{v}^{n+1,\nu}) &= 0 \, ,
\end{aligned}$$

solve for $\phi^{n+1,\nu}$, $\mathbf{v}^{n+1,\nu}$ and $\theta'|^{n+1,\nu}$ via elliptic problem for $\phi^{n+1,\nu}$;

$$\mathbf{B}_{\rm i}^{n+1,\nu-3/4} = \widehat{\mathbf{B}}_{\rm i} + 0.5\delta t \left[(\mathbf{B}^{n+1,\nu-3/4} \cdot \nabla)\mathbf{v}^{n+1,\nu} - \mathbf{B}^{n+1,\nu-3/4} (\nabla \cdot \mathbf{v}^{n+1,\nu}) \right]_{\rm i} \; ;$$

$$\begin{aligned}
\mathbf{B}_{\rm i}^{n+1,\nu} &= \widehat{\mathbf{B}}_{\rm i} + 0.5\delta t \left[(\mathbf{B}^{n+1,\nu-3/4} \cdot \nabla)\mathbf{v}^{n+1,\nu} - \mathbf{B}^{n+1,\nu-3/4} (\nabla \cdot \mathbf{v}^{n+1,\nu}) \right]_{\rm i} \\
&\quad -0.5\delta t \nabla\phi^*|^{n+1,\nu} \; , \\
\nabla \cdot \mathbf{B}^{n+1,\nu} &= 0 \, ,
\end{aligned}$$

solve for $\phi^*|^{n+1,\nu}$ and $\mathbf{B}^{n+1,\nu}$ via elliptic problem for $\phi^*|^{n+1,\nu}$.

Other examples include moist, Durran and compressible Euler equations.
Designing principles are always the same:

$$\frac{\partial \Phi}{\partial t} + \nabla \bullet (\mathbf{V}\Phi) = \mathbf{R} \ ,$$

$$\forall_i \quad \mathbf{\Phi}_i^{n+1} = \mathbf{\Phi}_i^{*} + 0.5\delta t \mathbf{R}_i^{n+1} \qquad \mathbf{\Phi}^{*} \equiv \mathcal{A}(\mathbf{\Phi}^{n} + 0.5\delta t \mathbf{R}^{n}, \widehat{\mathbf{V}}^{n+1/2})$$

$$\forall_i \quad \mathbf{\Phi}_i^{n+1,\ \mu} = \mathbf{\Phi}_i^{*} + 0.5\delta t \mathbf{R}_i^{n+1,\ \mu-1}$$

$$\begin{aligned} \| \, \mathbf{\Phi}^{n+1,\ \mu} - \mathbf{\Phi}^{n+1} \, \| \; &= \; 0.5\delta t \, \| \, \mathbf{R}(\mathbf{\Phi}^{n+1,\ \mu-1}) - \mathbf{R}(\mathbf{\Phi}^{n+1}) \, \| \\ &\leq 0.5\delta t \ \sup \| \, \partial \mathbf{R}/\partial \mathbf{\Phi} \, \| \| \, \mathbf{\Phi}^{n+1,\ \mu-1} - \mathbf{\Phi}^{n+1} \, \| \end{aligned}$$

Dynamic grid adaptivity

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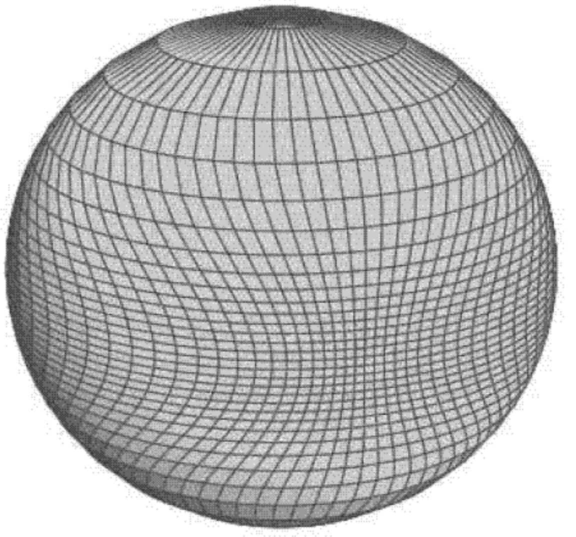
Prusa & Sm., *JCP* 2003; Wedi & Sm., *JCP* 2004, Sm. & Prusa, *IJNMF* 2005

- A generalized mathematical framework for the implementation of deformable coordinates in a generic Eulerian/semi-Lagrangian format of nonoscillatory-forward-in-time (NFT) schemes
- Technical apparatus of the Riemannian Geometry must be applied judiciously, in order to arrive at an effective numerical model.

Diffeomorphic mapping

$$(\overline{t}, \overline{x}, \overline{y}, \overline{z}) \equiv (\ t,\ E(t,x,y),\ D(t,x,y),\ C(t,x,y,z)\ )$$

(t,x,y,z) does not have to be Cartesian!



Example: Continuous global mesh transformation

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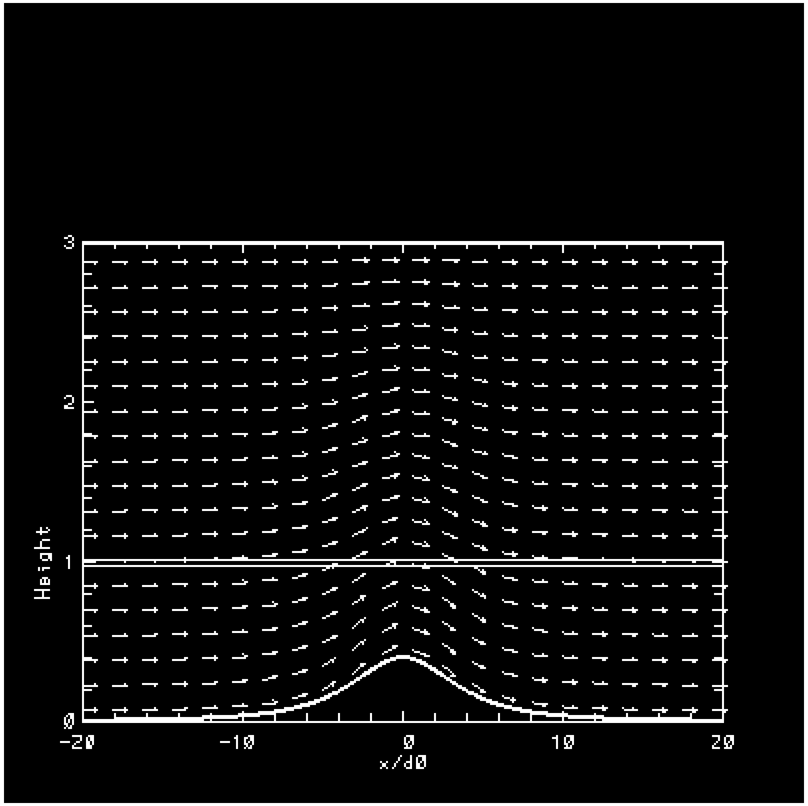
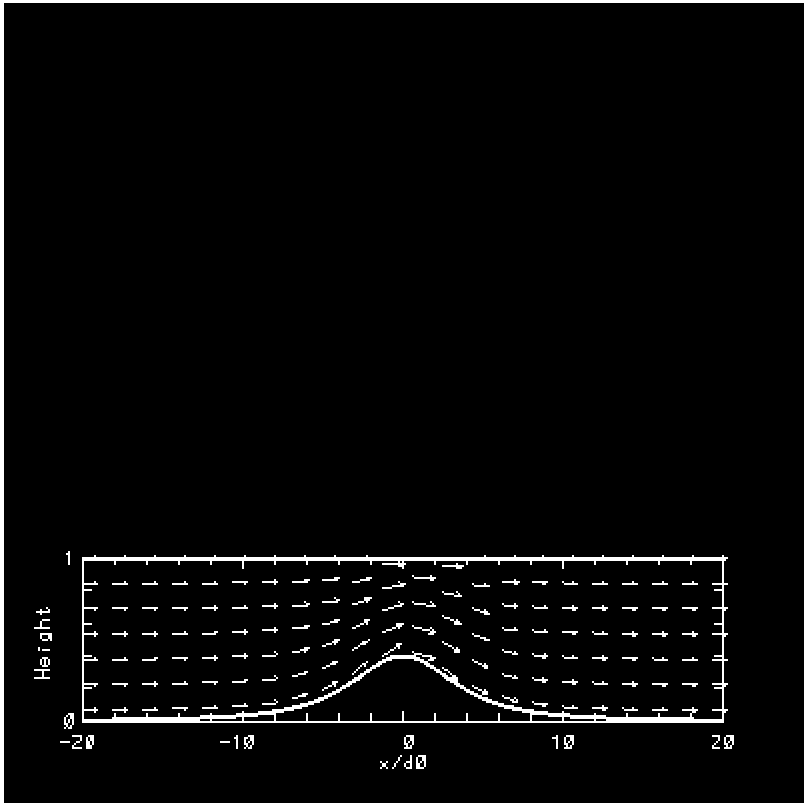
Example of free surface in anelastic model (Wedi & Sm., *JCP*,2004)





Example of IMB (Urban PBL, Smolarkiewicz et al. 2007, *JCP*)

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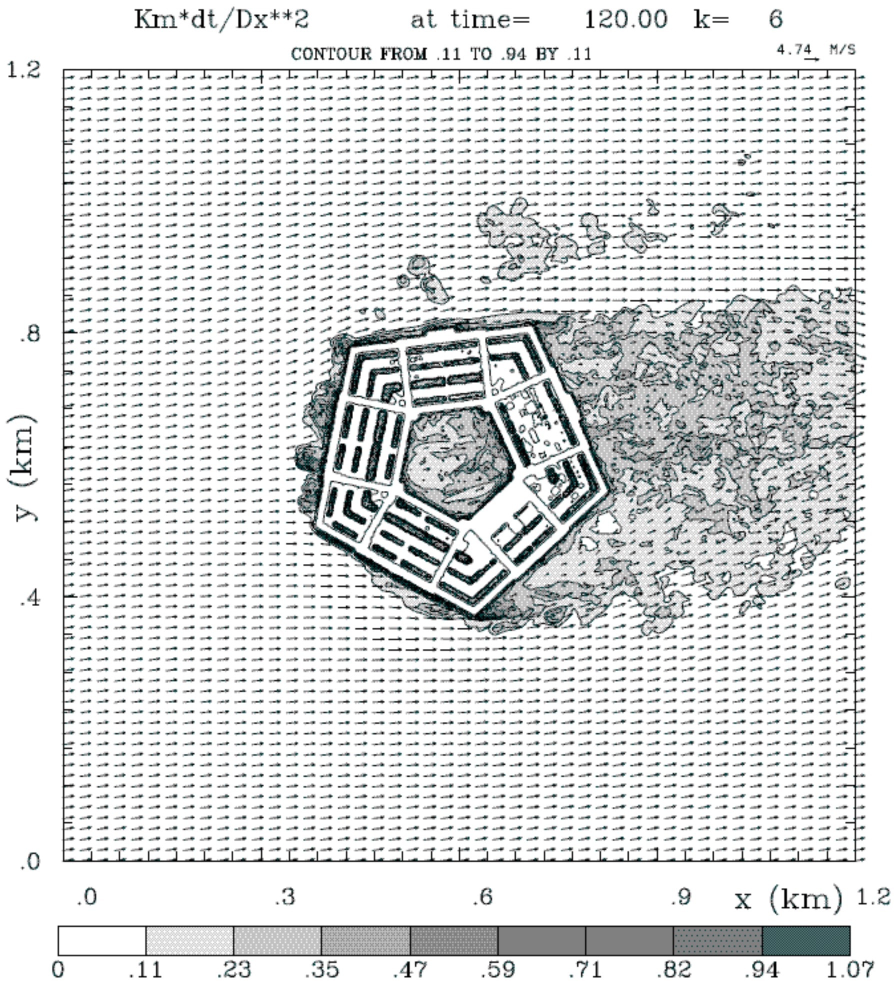

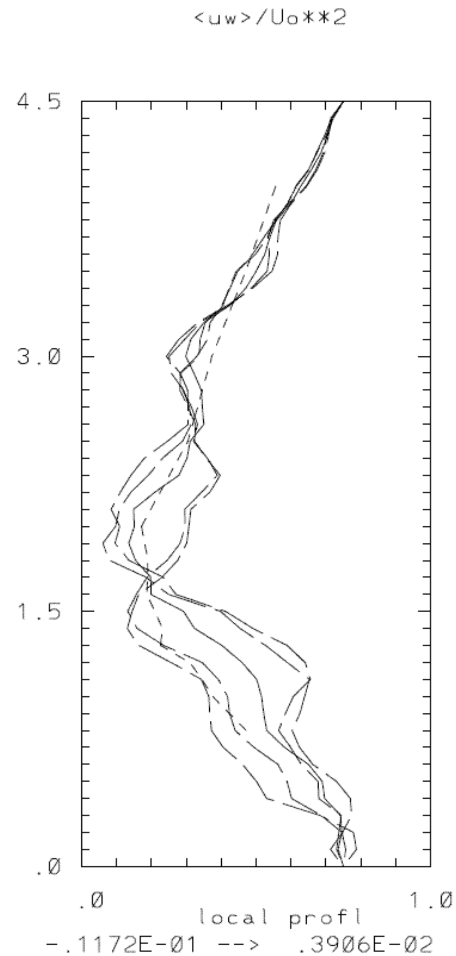

$\sqrt{TKE}$ contours in cross section at z=10 m

normalized profiles at a location in the wake $<u'w'>$

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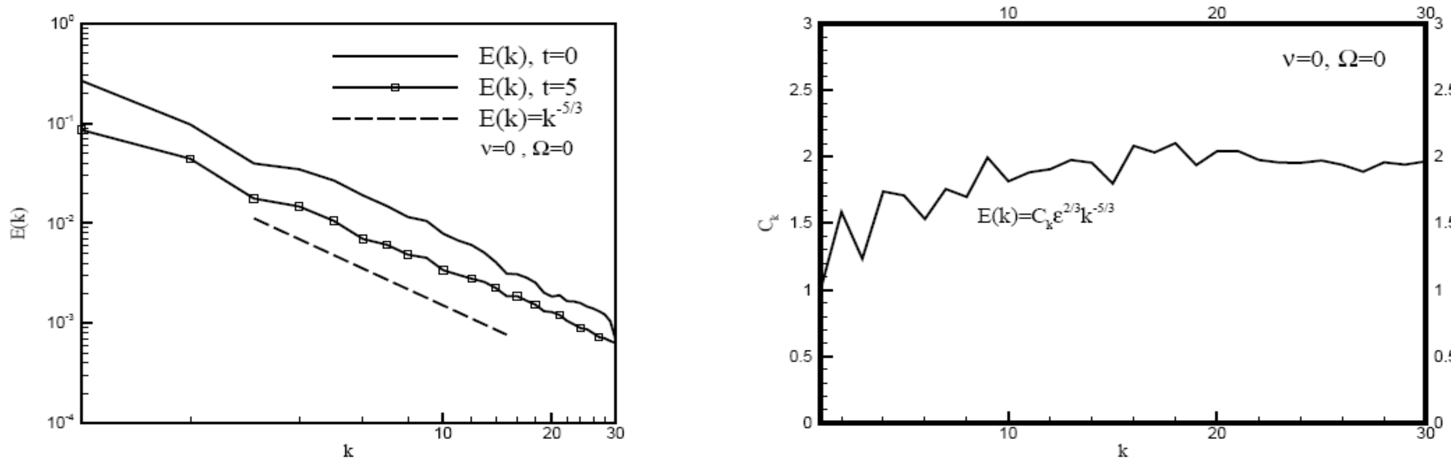
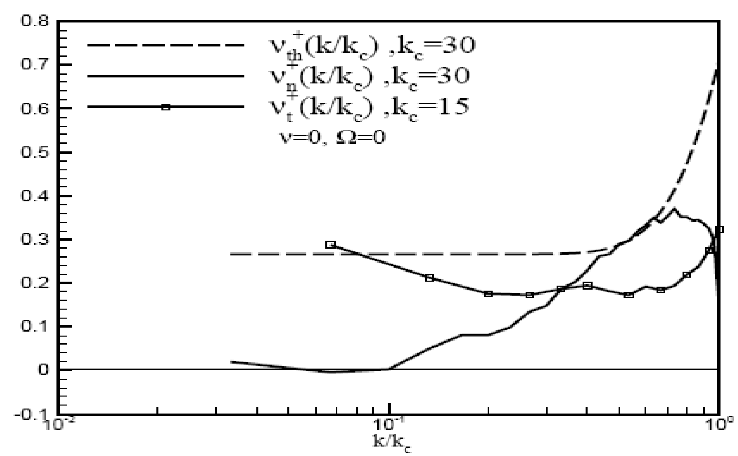
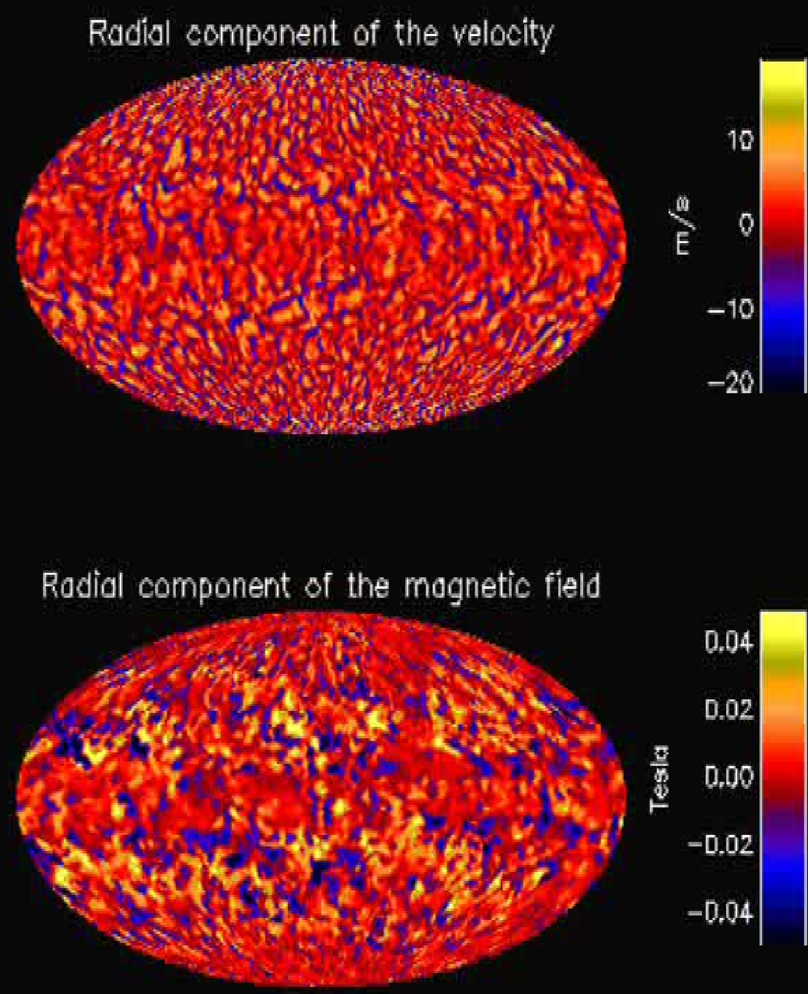
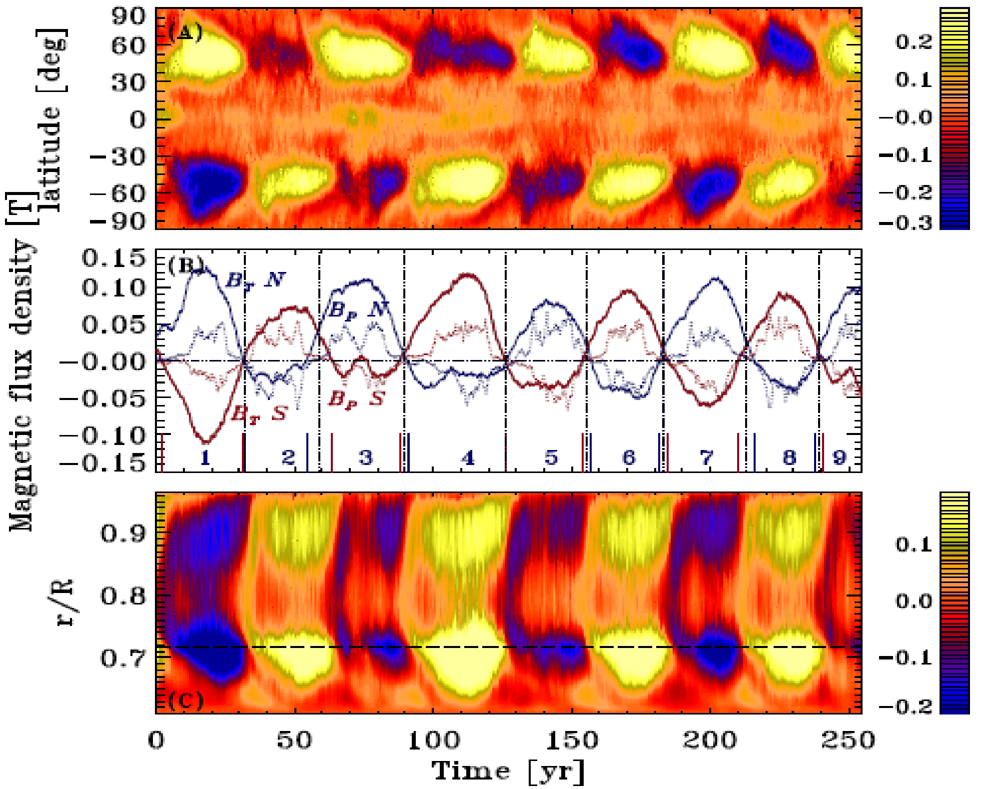


Figure 4: 64^3 ILES of decaying turbulence, Domaradzki et al. *Phys. Fluids* 2003. Energy spectra and Kolmogorov function $C_K(k) = \varepsilon^{-2/3}k^{5/3}E(k)$ dla $\nu = 0.0$ $\Leftrightarrow \ \langle(\delta v_{\|}(l))^2\rangle \sim l^{2/3}$

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu k^2 E(k) - \varepsilon_n(k,t) \Rightarrow \varepsilon_n := 2\nu_n k^2 E(k) \Rightarrow \nu_n(k)$$







Toroidal component of **B** in the uppermost portion of the stable layer underlying the convective envelope at r/R≈0 .7 ➔