A low Mach number model for moist convection

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Goal

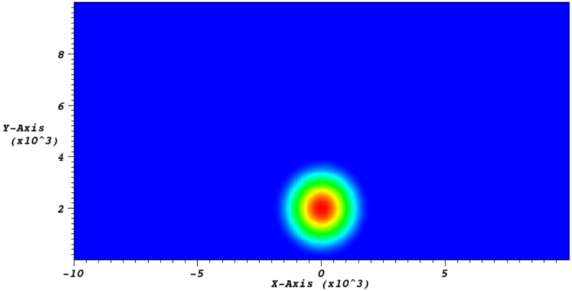

Figure 1: initial conditions of the potential temperature

Successfully model a hot rising bubble in a moist atmosphere with

- phase changes
- latent heat
- sound waves removed

Why pseudo-incompressible?

- filters sound waves allowing longer time steps
- advantages over the anaelastic approximation:
  - more easily extended to compressible equations
  - allows higher variation in ρ and θ
  - more accurate for small scales e.g. combustion
- P-I code currently being developed

The Pseudo-Incompressible approximation

The Compressible Euler Equations

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$
$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \circ \mathbf{u}) + \nabla p = -g\mathbf{k}$$
$$(\rho\theta)_t + \nabla \cdot (\rho\theta \mathbf{u}) = 0$$

Equations of state

$$\theta = T \left(\frac{p_{ref}}{p} \right)^{R/c_p}, \quad p = \rho R T$$

or, combining these equations

$$\rho\theta = \frac{p_{ref}}{R} \left(\frac{p}{p_{ref}} \right)^{c_v/c_p} \tag{1}$$

The Pseudo-Incompressible approximation

Assume

$$p = p_0(z) + p'(\mathbf{x}, t)$$
$$\rho = \rho_0(z) + \rho'(\mathbf{x}, t)$$

where $\frac{\partial p_0}{\partial z} = -\rho_0 g$ and $p'/p_0 << 1$.

Now set $p = p_0$ in (1) and let $\gamma = c_p/c_v$

$$\rho^* \theta = \frac{p_{ref}}{R} \left(\frac{p_0}{p_{ref}} \right)^{1/\gamma}$$

where $\rho^*(p_0, \theta)$ is called the pseudo-density.

The Pseudo-Incompressible approximation

The conservation of potential temperature equation becomes a divergence constraint

$$(\rho^*\theta)_t + \nabla \cdot (\rho^*\theta \mathbf{u}) = \nabla \cdot (\rho^*\theta \mathbf{u}) = 0$$

and the governing equations become

$$\begin{aligned}
&\rho^*_t + \nabla \cdot (\rho^* \mathbf{u}) = 0 \\
&(\rho^* \mathbf{u})_t + \nabla \cdot (\rho^* \mathbf{u} \circ \mathbf{u}) + \nabla p = -\rho^* g \mathbf{k} \\
&\nabla \cdot (\rho^* \theta \mathbf{u}) = 0
\end{aligned}$$

with equation of state

$$\rho^*\theta = \frac{p_{ref}}{R} \left(\frac{p_0}{p_{ref}} \right)^{1/\gamma}$$

Adding moisture

Assumptions

- each state has the same temperature and velocity field
- using a simplified EOS
- ignoring: precipitation, ice-phase microphysics, Coriolis force, subgrid-scale turbulence

Adding moisture

Moist compressible equations with bulk thermodynamics

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$
$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \circ \mathbf{u}) + \nabla p = -g\mathbf{k}$$
$$(\rho\theta)_t + \nabla \cdot (\rho\theta \mathbf{u}) = \rho\theta S$$
$$\frac{Dr_v}{Dt} = \dot{r}_{cond}, \quad \frac{Dr_c}{Dt} = -\dot{r}_{cond}$$

where $\rho = (\rho_a + \rho_v + \rho_c)$, $r_v = \rho_v/\rho_a$ and $r_c = \rho_c/\rho_a$.

Equation of state

$$\rho\theta = \frac{p_{ref}}{R}\left(\frac{p}{p_{ref}}\right)^{1/\gamma}$$

First (Naive) Attempt

If we proceed as in the case without moisture the potential temperature equation becomes

$$\nabla \cdot (\rho^* \theta \mathbf{u}) = \rho^* \theta S$$

This does NOT work with solid wall boundary conditions.

Second attempt

Using a time varying background pressure $p_0(z, t)$

$$(\rho^*\theta)_t = \frac{p_{ref}}{R}\frac{\partial}{\partial t}\left(\left(\frac{p_0}{p_{ref}}\right)^{1/\gamma}\right) = \frac{\rho^*\theta}{\gamma p_0}(p_0)_t$$

$$\Rightarrow \nabla \cdot (\rho^*\theta\mathbf{u}) = \rho^*\theta\left(S - \frac{1}{\gamma p_0}(p_0)_t\right)$$

The problem now is how do we calculate $(p_0)_t$?

Calculating the time varying background state

From the work of Almgren et al on Supernovae:

- let $w_0(z,t)$ be the vertical velocity field that adjusts the base state and let $\tilde{\mathbf{u}}$ govern the remaining local dynamics

$$\mathbf{u} = w_0\mathbf{k} + \tilde{\mathbf{u}}$$

  where $\int_{x_{min}}^{x_{max}} \tilde{w}\,\mathrm{d}x = 0$.

- assume the background pressure of each parcel remains unchanged.

$$\frac{Dp_0}{Dt} = \frac{\partial p_0}{\partial t} + w_0\frac{\partial p_0}{\partial z} = 0$$

  $\Rightarrow$ can no longer use a solid top wall. We must compromise and use a buffer layer.

Calculating the time varying background state

Writing the divergence constraint in terms of w_0 and $\tilde{\mathbf{u}}$

$$\nabla \cdot (\rho^* \theta w_0 \mathbf{k}) + \nabla \cdot (\rho^* \theta \tilde{\mathbf{u}}) = \rho^* \theta \left(S - \frac{1}{\gamma p_0} (p_0)_t \right)$$

Integrating over a horizontal slab $[x_{min}, x_{max}] x [z - h, z + h]$

$$\int\limits_{z-h}^{z+h} \int\limits_{x_{min}}^{x_{max}} \left[\nabla \cdot (\rho^* \theta w_0 \mathbf{k}) + \nabla \cdot (\rho^* \theta \tilde{\mathbf{u}}) \right] \mathrm{d}z \mathrm{d}x$$

$$= \int\limits_{z-h}^{z+h} \int\limits_{x_{min}}^{x_{max}} \left[\rho^* \theta \left(S - \frac{1}{\gamma p_0} (p_0)_t \right) \right] \mathrm{d}z \mathrm{d}x$$

Calculating the time varying background state

Assuming solid horizontal walls

$$\int_{x_{min}}^{x_{max}} \left[(\rho^*\theta w_0) + (\rho^*\theta \tilde{w})\right]\Big|_{z-h}^{z+h} \mathrm{d}x = \int_{z-h}^{z+h}\int_{x_{min}}^{x_{max}} \left[\rho^*\theta\left(S - \frac{1}{\gamma p_0}(p_0)_t\right)\right] \mathrm{d}z\mathrm{d}x$$
$$= L\int_{z-h}^{z+h} \left[\rho^*\theta\left(\overline{S} - \frac{1}{\gamma p_0}(p_0)_t\right)\right] \mathrm{d}z \qquad (2)$$

where $\overline{S}(z,t) = \frac{1}{L}\int_{x_{min}}^{x_{max}} S(x,z,t)\,\mathrm{d}x$ and we've used the fact that $\rho^*\theta$ does not depend x.

Calculating the time varying background state

Using the definition of $\tilde{w}$ and the fact that $(\rho^* \theta w_0)$ does not depend on x, (2) becomes

$$L(\rho^* \theta w_0)\Big|_{z-h}^{z+h} = L \int_{z-h}^{z+h} \left[\rho^* \theta \left(\overline{S} - \frac{1}{\gamma p_0}(p_0)_t \right) \right] \mathrm{d}z$$

Cancelling L, dividing by h and taking the limit $h \to 0$

$$\frac{\partial \rho^* \theta w_0}{\partial z} = \rho^* \theta \left[\overline{S} - \frac{1}{\gamma p_0} \frac{\partial p_0}{\partial t} \right] \tag{3}$$

Calculating the time varying background state

Expanding (3)

$$\begin{aligned} w_0 \frac{\partial \rho^* \theta}{\partial z} + \rho^* \theta \frac{\partial w_0}{\partial z} &= w_0 \frac{\rho^* \theta}{\gamma p_0} \frac{\partial p_0}{\partial z} + \rho^* \theta \frac{\partial w_0}{\partial z} \\ &= \rho^* \theta \left[\overline{S} - \frac{1}{\gamma p_0} \left(-w_0 \frac{\partial p_0}{\partial z} \right) \right] \end{aligned}$$

Cancelling the $w_0 \frac{\rho^* \theta}{\gamma p_0} \frac{\partial p_0}{\partial z}$ terms

$$\begin{aligned} &\Rightarrow \rho^* \theta \frac{\partial w_0}{\partial z} = \rho^* \theta \overline{S} \\ &\Rightarrow \frac{\partial w_0}{\partial z} = \overline{S} \end{aligned}$$

Assuming solid walls at the base

$$w_0(z, t) = \int_0^z \overline{S}(z', t) \, \mathrm{d}z'$$

New divergence constraint

The background divergence constraint

$$\frac{\partial \rho^* \theta w_0}{\partial z} = \rho^* \theta \left[\overline{S} - \frac{1}{\gamma p_0} \frac{\partial p_0}{\partial t} \right]$$

The original divergence constraint

$$\nabla \cdot (\rho^* \theta \mathbf{u}) = \rho^* \theta \left(S - \frac{1}{\gamma p_0} (p_0)_t \right)$$

subtracting we get

$$\nabla \cdot (\rho^* \theta \mathbf{u}) = \rho^* \theta (S - \overline{S}) + \frac{\partial}{\partial z} (\rho^* \theta w_0)$$

New governing equations

Mass	$\rho^*_t + \nabla \cdot (\rho^* \mathbf{u}) = 0$
Momentum	$(\rho^* \mathbf{u})_t + \nabla \cdot (\rho^* \mathbf{u} \circ \mathbf{u})) + \nabla p = -\rho^* g \mathbf{k}$
Divergence constraint	$\nabla \cdot (\rho^* \theta \mathbf{u}) = \rho^* \theta (S - \overline{S}) + \frac{\partial}{\partial z}(\rho^* \theta w_0)$
Microphysics	$\frac{Dr_v}{Dt} = \dot{r}_{cond}, \quad \frac{Dr_c}{Dt} = -\dot{r}_{cond}$
Equation of state	$\rho\theta = \frac{p_{ref}}{R} \left(\frac{p_0}{p_{ref}} \right)^{1/\gamma}$
Base-state updates	$\begin{cases} w_0(z,t) = \int_0^z \overline{S}(z', t)\, \mathrm{d}z' \\ \frac{\partial p_0}{\partial t} + w_0 \frac{\partial p_0}{\partial z} = 0 \end{cases}$

Currently

- implementing model in “in-house” low mach number finite volume code
- using Grabowski & Smolarkiewicz 1990 for the source terms
- having problems with the projection steps which enforce the divergence constraint

Eventually

- add precipitation
- more complicated thermodyanmics and microphysics
- parallelisation and mesh-refinement (already implemented in dry case)
- other test cases e.g. the more realistic test-case of Klassen & Clark 1985, squall-lines....

Thank you for your attention