Perturbation equations for all-scale atmospheric dynamics

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Aim: Extended predictability of the whole weather active atmosphere

Premise: The atmospheric dynamics constitutes a relatively small perturbation about dominant balances of hydrostacy and geostrophy established in effect of the Earth gravity, rotation, stably stratified thermal structure of its atmosphere and the incoming flux of solar energy.

Inference: Given the approximate nature of the available solution methods, it is compelling to formulate governing PDEs in terms of perturbation variables about an arbitrary state of the atmosphere that already satisfies some or all of the dominant balances.

Formulating governing PDEs in terms of perturbation variables is well known in small-scale atmospheric dynamics (Boussinesq, 1903), where hydrostatically balanced *reference* states are used to justify filtering out energetically insignificant acoustic modes from the compressible Euler equations under gravity.

Our notion of *ambient* states is distinct. The role of ambient states is to enhance the efficacy of numerical simulation; e.g. by simplifying the initial and boundary conditions and/or improving the conditioning of elliptic BVPs, without any system linearization ***!***

The key assumption is that the ambient state is a particular solution of the governing problem, so that subtracting its own minimal set of PDEs from the governing equations can provide a useful perturbation form of the governing system.

Ambient states are not limited to stable or neutral stratifications, can be spatially and temporally varying to represent, e.g., thermally balanced large-scale steady flows in atmospheric models or prescribe oceanic tidal motions.

Here, we consider a generalized formalism allowing, in principle, for arbitrary ambient state. *All derived PDEs are mathematically equivalent but lead to different solvers* ***!***

Governing equations

Generic form

$$\frac{d\rho}{dt} = -\frac{\rho}{\mathcal{G}}\nabla\cdot(\mathcal{G}\mathbf{v}) \ ,$$

$$\frac{d\theta}{dt} = \mathcal{H} \ ,$$

$$\frac{d\mathbf{u}}{dt} = -\frac{\theta}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi + \mathbf{g} - \mathbf{f}\times\mathbf{u} + \boldsymbol{\mathcal{M}}(\mathbf{u}) + \boldsymbol{\mathcal{D}}$$

$$\nabla = (\partial_x,\, \partial_y,\, \partial_z)$$

$$d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$$

$$\mathbf{v} = \widetilde{\mathbf{G}}^T\mathbf{u} ,\ \mathcal{G}$$

$$\phi = c_p\theta_0\pi$$

$$\phi = c_p\theta_0\left[\left(\frac{R}{p_0}\rho\theta\right)^{R/c_v}\right]$$

Ambient state $(\rho_a, \theta_a, \phi_a, \mathbf{u}_a)$

$$\frac{d_a\rho_a}{dt} = -\frac{\rho_a}{\mathcal{G}}\nabla\cdot\mathcal{G}\mathbf{v}_a \ ,$$

$$\frac{d_a\theta_a}{dt} = \mathcal{H}_a \ ,$$

$$\frac{d_a\mathbf{u}_a}{dt} = -\frac{\theta_a}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi_a + \mathbf{g} - \mathbf{f}\times\mathbf{u}_a + \boldsymbol{\mathcal{M}}(\mathbf{u}_a) + \boldsymbol{\mathcal{D}}(\mathbf{u}_a)$$

$$d_a/dt = \partial/\partial t + \mathbf{v}_a\cdot\nabla$$

$$\mathbf{v}_a = \widetilde{\mathbf{G}}^T\mathbf{u}_a$$

$$\phi_a = c_p\theta_0\pi_a$$

$$\phi_a = c_p\theta_0\left[\left(\frac{R}{p_0}\rho_a\theta_a\right)^{R/c_v}\right]$$

Perturbation forms

Definitions: $\theta' := \theta - \theta_a \;,\quad \mathbf{u}' := \mathbf{u} - \mathbf{u}_a \;,\quad \phi' := \phi - \phi_a$

Key auxiliary relations: $\forall\ \psi = \psi' + \psi_a \;,\quad \frac{d\psi}{dt} - \frac{d_a\psi_a}{dt} \equiv \frac{d\psi'}{dt} + \mathbf{v}' \cdot \nabla\psi_a \;;$

$$\frac{\theta}{\theta_0}\widetilde{\mathrm{G}}\nabla\phi - \frac{\theta_a}{\theta_0}\widetilde{\mathrm{G}}\nabla\phi_a \equiv \frac{\theta}{\theta_0}\widetilde{\mathrm{G}}\nabla\phi' + \frac{\theta'}{\theta_0}\widetilde{\mathrm{G}}\nabla\phi_a \;.$$

Then, subtracting ambient from generic PDEs ➔

$$\frac{d\theta'}{dt} = -\mathbf{v}' \cdot \nabla\theta_a + \mathcal{H}' \;,$$

$$\frac{d\mathbf{u}'}{dt} = -\mathbf{v}' \cdot \nabla\mathbf{u}_a - \frac{\theta}{\theta_0}\widetilde{\mathrm{G}}\nabla\phi' - \frac{\theta'}{\theta_0}\widetilde{\mathrm{G}}\nabla\phi_a - \mathbf{f} \times \mathbf{u}' + \boldsymbol{\mathcal{M}}'(\mathbf{u}, \mathbf{u}_a) + \boldsymbol{\mathcal{D}}'(\mathbf{u}, \mathbf{u}_a)$$

GBIS

where: $\mathcal{H}' = \mathcal{H} - \mathcal{H}_a \;,$

$\mathbf{v}' = \widetilde{\mathrm{G}}^T\mathbf{u}' \;,$

$\boldsymbol{\mathcal{M}}'(\mathbf{u}, \mathbf{u}_a) = \boldsymbol{\mathcal{M}}(\mathbf{u}' + \mathbf{u}_a) - \boldsymbol{\mathcal{M}}(\mathbf{u}_a) \;,$

$\boldsymbol{\mathcal{D}}'(\mathbf{u}, \mathbf{u}_a) = \boldsymbol{\mathcal{D}}(\mathbf{u}' + \mathbf{u}_a) - \boldsymbol{\mathcal{D}}(\mathbf{u}_a) \;,$ ➔ $\boldsymbol{\mathcal{D}}' = \boldsymbol{\mathcal{D}}(\mathbf{u}')$ *for flow independent viscosity*

Discussion

implicit, centred advection of the ambient state

$$\frac{d\theta'}{dt} = -\mathbf{v}' \cdot \nabla\theta_a + \mathcal{H}' \ ,$$

3D buoyancy

$$\frac{d\mathbf{u}'}{dt} = -\mathbf{v}' \cdot \nabla \mathbf{u}_a - \frac{\theta}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi' - \frac{\theta'}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi_a$$
$$-\mathbf{f}\times\mathbf{u}' + \boldsymbol{\mathcal{M}}'(\mathbf{u},\mathbf{u}_a) + \boldsymbol{\mathcal{D}}'(\mathbf{u},\mathbf{u}_a)$$

GBIS

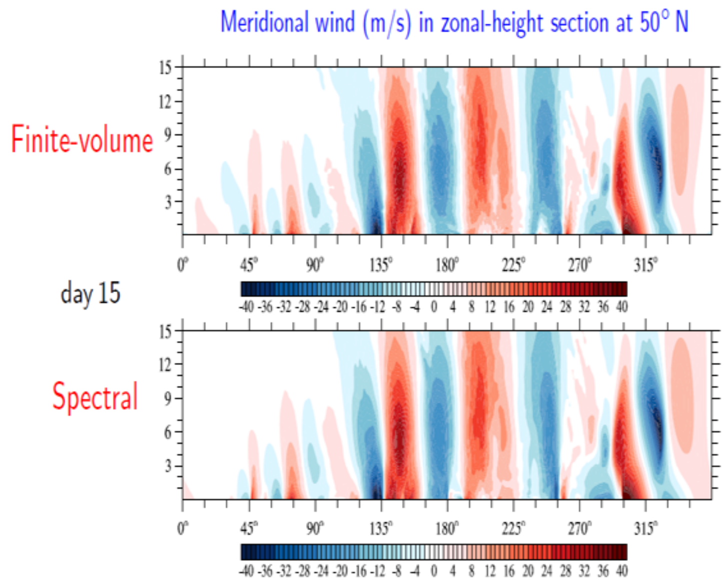
$$\widetilde{\mathbf{G}}\nabla\phi_a = \frac{\theta_0}{\theta_a}\left(\mathbf{g} - \mathbf{f}\times\mathbf{u}_a + \boldsymbol{\mathcal{M}}(\mathbf{u}_a) + \boldsymbol{\mathcal{D}}(\mathbf{u}_a) - \frac{d_a\mathbf{u}_a}{dt}\right)$$

$$\frac{d\mathbf{u}}{dt} = -\frac{\theta}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi' - \mathbf{g}\frac{\theta'}{\theta_a} - \mathbf{f}\times\left(\mathbf{u} - \frac{\theta}{\theta_a}\mathbf{u}_a\right)$$
$$+\left(\boldsymbol{\mathcal{M}}(\mathbf{u}) - \frac{\theta}{\theta_a}\boldsymbol{\mathcal{M}}(\mathbf{u}_a)\right) + \left(\boldsymbol{\mathcal{D}}(\mathbf{u}) - \frac{\theta}{\theta_a}\boldsymbol{\mathcal{D}}(\mathbf{u}_a)\right) + \frac{\theta}{\theta_a}\frac{d_a\mathbf{u}_a}{dt}$$

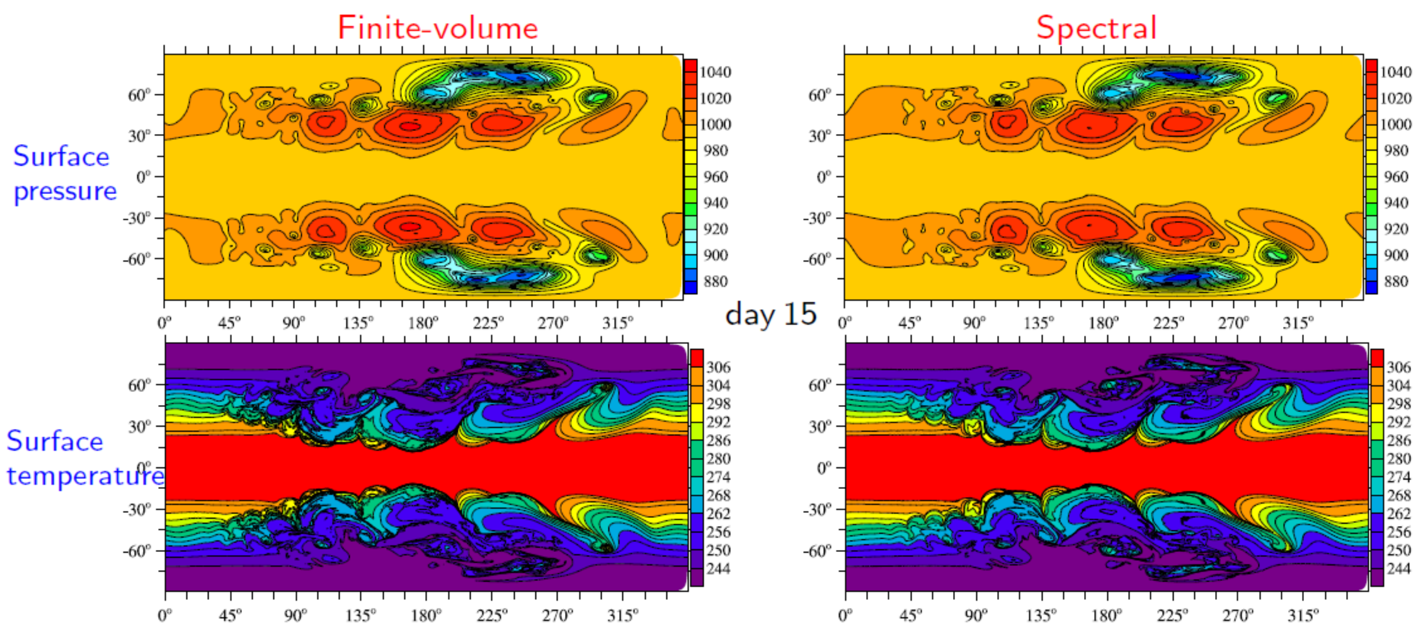
REF

currently employed in the Finite-Volume Module (FVM) of the Integrated Forecasting System (IFS)

FVM (REF) & IFS



Dry baroclinic instability, FVM (O640) versus the spectral IFS ($T_{co}639$):



Numerical solutions

Conservation form of perturbational PDEs (GBIS):

$$\frac{\partial \mathcal{G}\rho}{\partial t} + \nabla \cdot (\mathcal{G}\rho \mathbf{v}) = 0 \ ,$$

$$\frac{\partial \mathcal{G}\rho\theta'}{\partial t} + \nabla \cdot (\mathcal{G}\rho \mathbf{v}\theta') = -\mathcal{G}\rho\left(\widetilde{\mathbf{G}}^T\mathbf{u}' \cdot \nabla\theta_a - \mathcal{H}' - \alpha^\theta\theta'\right) \ ,$$

$$\frac{\partial \mathcal{G}\rho\mathbf{u}'}{\partial t} + \nabla \cdot (\mathcal{G}\rho \mathbf{v} \otimes \mathbf{u}') = -\,\mathcal{G}\rho\Big(\widetilde{\mathbf{G}}^T\mathbf{u}' \cdot \nabla\mathbf{u}_a + \frac{\theta}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi' + \frac{\theta'}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi_a + \mathbf{f} \times \mathbf{u}' - \mathcal{M}'(\mathbf{u},\mathbf{u}_a) - \mathcal{D}'(\mathbf{u},\mathbf{u}_a) - \alpha^u\mathbf{u}'\Big)$$

Template NFT algorithm:

$$\boldsymbol{\psi}_{\mathrm{i}}^{n+1} = \mathcal{A}_{\mathrm{i}}\left(\boldsymbol{\psi}^n + 0.5\delta t\,\boldsymbol{\mathcal{R}}(\boldsymbol{\psi}^n)\right) + 0.5\delta t\,\boldsymbol{\mathcal{R}}_{\mathrm{i}}(\boldsymbol{\psi}^{n+1}) \ \equiv \ \widehat{\boldsymbol{\psi}}_{\mathrm{i}} + 0.5\delta t\,\boldsymbol{\mathcal{R}}_{\mathrm{i}}(\boldsymbol{\psi}^{n+1})$$

Step 1: mass continuity & advectors for specific variables

$$\rho_{\mathrm{i}}^{n+1} = \mathcal{A}_{\mathrm{i}}\left(\rho^n, (\mathbf{v}\mathcal{G})^{n+1/2}, \mathcal{G}^n, \mathcal{G}^{n+1}\right) \implies \mathbf{V}^{n+1/2} = \overline{\mathbf{v}^{\perp}\mathcal{G}\rho}^{n+1/2}$$

Step 2: advecting specific variables & formulating semi-implicit solver

$$\theta'^{\,n+1}_{\mathrm{i}} = \widehat{\theta'}_{\mathrm{i}} - 0.5\delta t\left(\widetilde{\mathbf{G}}^T \mathbf{u}'^{\,n+1}\cdot\nabla\theta_a^{n+1} + \alpha^{\theta}\theta'^{\,n+1}\right)_{\mathrm{i}}$$

$$\begin{aligned}\mathbf{u}'^{\,n+1}_{\mathrm{i}} = \widehat{\mathbf{u}'}_{\mathrm{i}} &- 0.5\delta t\left(\widetilde{\mathbf{G}}^T \mathbf{u}'^{\,n+1}\cdot\nabla\mathbf{u}_a^{n+1}\right)_{\mathrm{i}} \\ &- 0.5\delta t\left(\frac{\theta^{\star}}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi'^{\,n+1} + \frac{\theta'^{\,n+1}}{\theta_0}\widetilde{\mathbf{G}}\nabla\phi_a^{n+1}\right)_{\mathrm{i}} \\ &- 0.5\delta t\left(\mathbf{f}\times\mathbf{u}'^{\,n+1} - \mathcal{M}'(\mathbf{u}^{\star},\mathbf{u}_a^{n+1}) + \alpha^u\mathbf{u}'^{\,n+1}\right)_{\mathrm{i}}\end{aligned}$$

where

$$\widehat{\theta'}_{\mathrm{i}} = \mathcal{A}_{\mathrm{i}}\left(\widetilde{\theta'}, \mathbf{V}^{n+1/2}, (\rho\mathcal{G})^n, (\rho\mathcal{G})^{n+1}\right) \;,\quad \widetilde{\theta'} = \left(\theta' + 0.5\delta t\mathcal{R}^{\theta}\right)^n \;;$$

$$\widehat{\mathbf{u}'}_{\mathrm{i}} = \mathcal{A}_{\mathrm{i}}\left(\widetilde{\mathbf{u}'}, \mathbf{V}^{n+1/2}, (\rho\mathcal{G})^n, (\rho\mathcal{G})^{n+1}\right) \;,\quad \widetilde{\mathbf{u}'} = \left(\mathbf{u}' + 0.5\delta t\boldsymbol{\mathcal{R}}^{\mathrm{u}}\right)^n \;;$$

$$\mathbf{L}\,\mathbf{u}' = \widehat{\widehat{\mathbf{u}'}} - \tau^u \Theta^\star \,\widetilde{\nabla}\phi' \implies$$

$$\mathbf{u}' = \check{\check{\mathbf{u}}}' - \mathbf{C}\,\nabla\phi' \;; \quad \check{\check{\mathbf{u}}}' = \mathbf{L}^{-1}\,\widehat{\widehat{\mathbf{u}'}}\;, \quad \mathbf{C} = \tau^u \Theta^\star\,\mathbf{L}^{-1}\widetilde{\mathbf{G}}\;,$$

Given all entries of **L ➔ $\mathbf{L}^{-1}$ ➔ C,** and the rest is straightforward in the Finite-Volume Module of IFS: taking d/dt of the perturbation form of the gas law, and integrating it consistently with all other dependent variables leads to the elliptic Helmholtz equation for pressure perturbation, the solution of which essentially completes the time step.

$$0 = -\sum_{\ell=1}^{3}\left(\frac{A_\ell^\star}{\zeta_\ell}\nabla\cdot\zeta_\ell(\check{\check{\mathbf{v}}} - \widetilde{\mathbf{G}}^T\mathbf{C}\nabla\varphi)\right) - B^\star(\varphi - \widehat{\varphi})$$

Poisson BVP

$$\frac{1}{\zeta_\ell}\frac{\partial}{\partial x^j}\left[\zeta_\ell\mathscr{E}\left(\widetilde{\mathscr{V}}^j - \widetilde{\mathscr{C}}^{jk}\frac{\partial\varphi}{\partial x^k}\right)\right] = 0\;,$$

a common molecule

Results

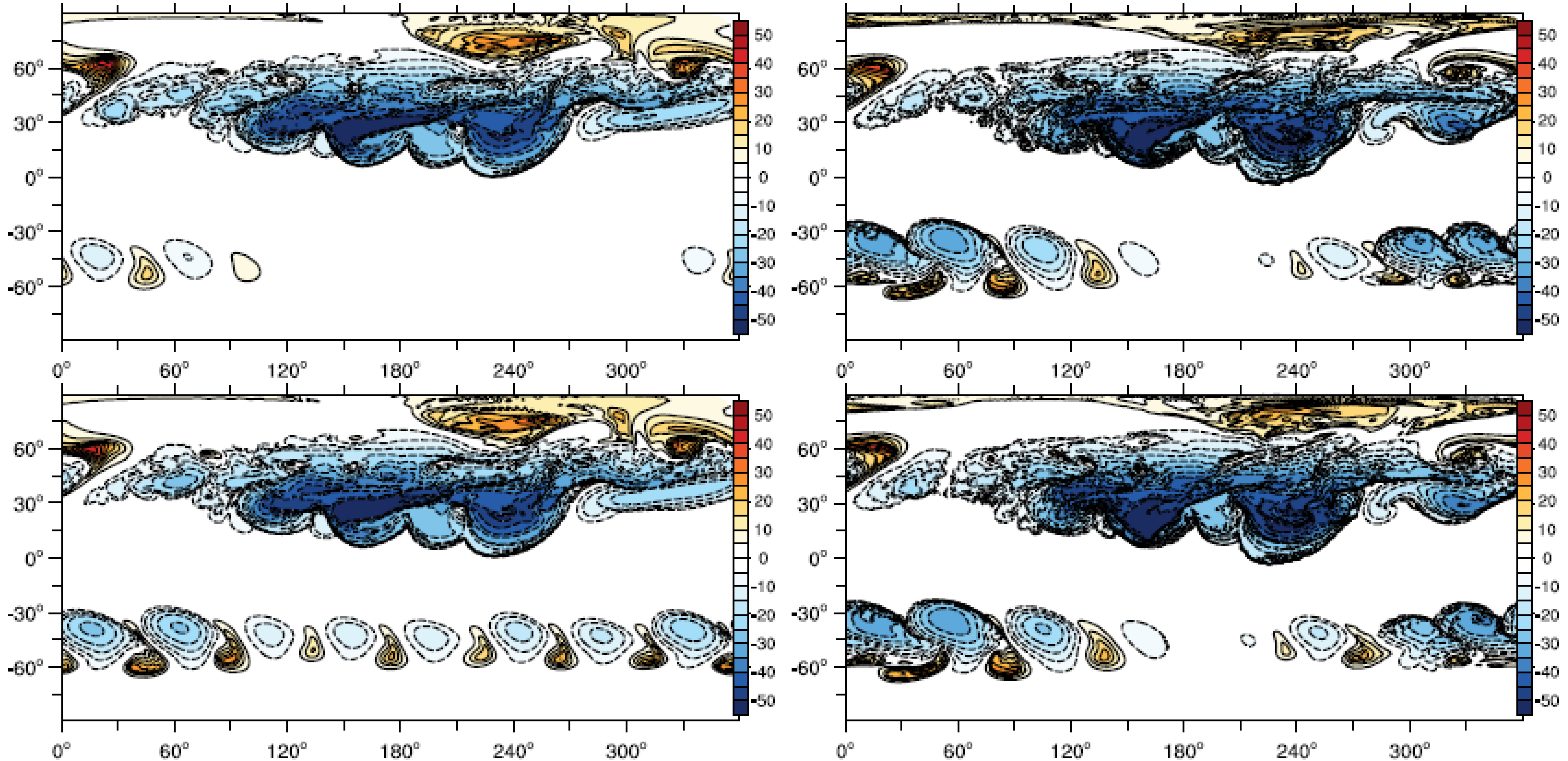


Fig. 1. Surface potential temperature perturbation θ' after 18 simulated days. The solutions GBIS and REF, corresponding to perturbation equations (7b) and (11) are shown at the top and bottom, respectively. The results from the O180 and 0640 mesh are shown on the left and right, correspondingly.

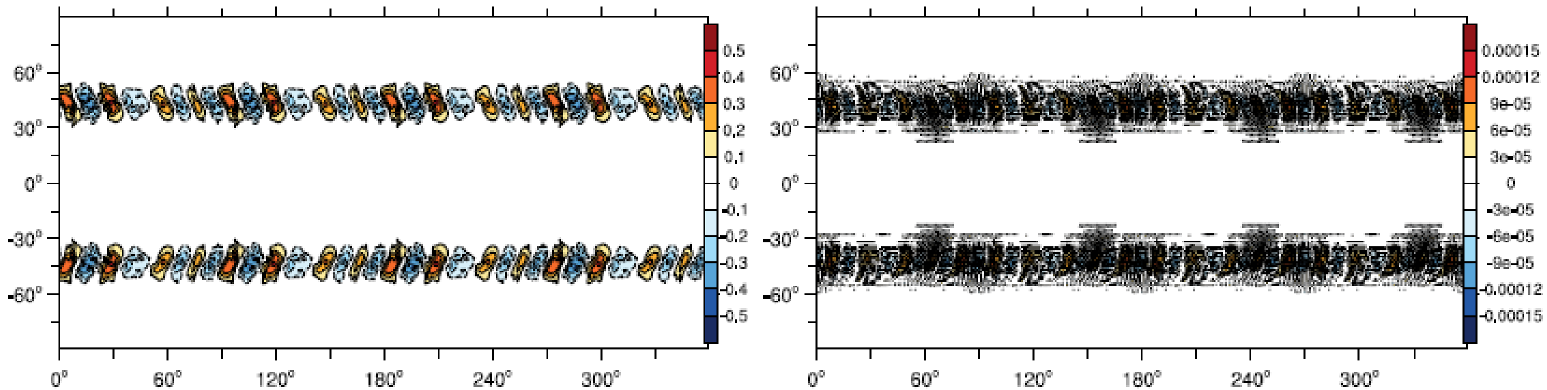


Fig. 2. The 18 days surface θ' REF (left) and GBIS (right) solutions on the O180 grid without the initial perturbation.

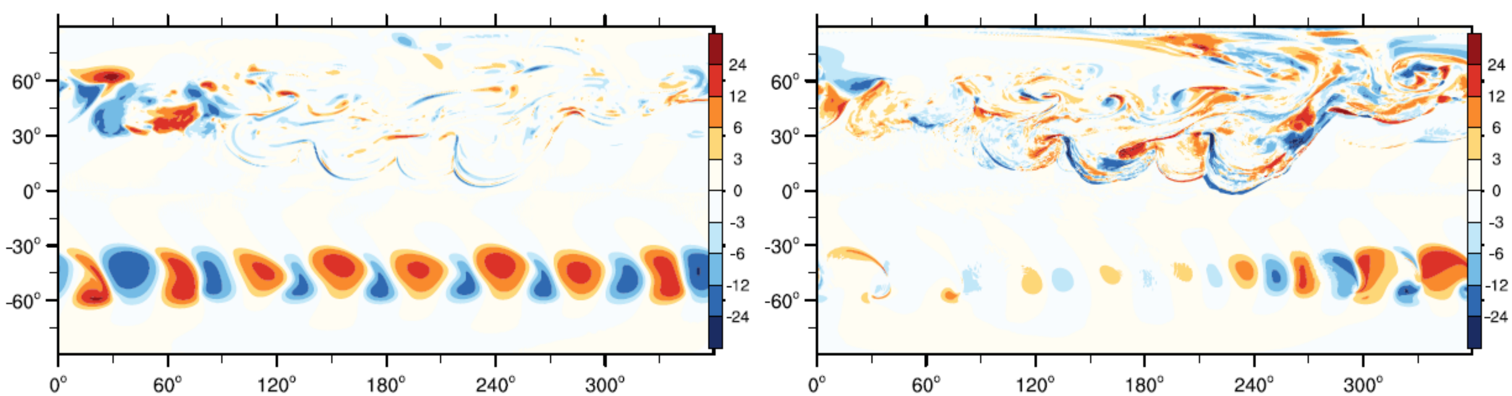


Fig. 3. GBIS-REF differences of the θ' solutions in Fig. 1.

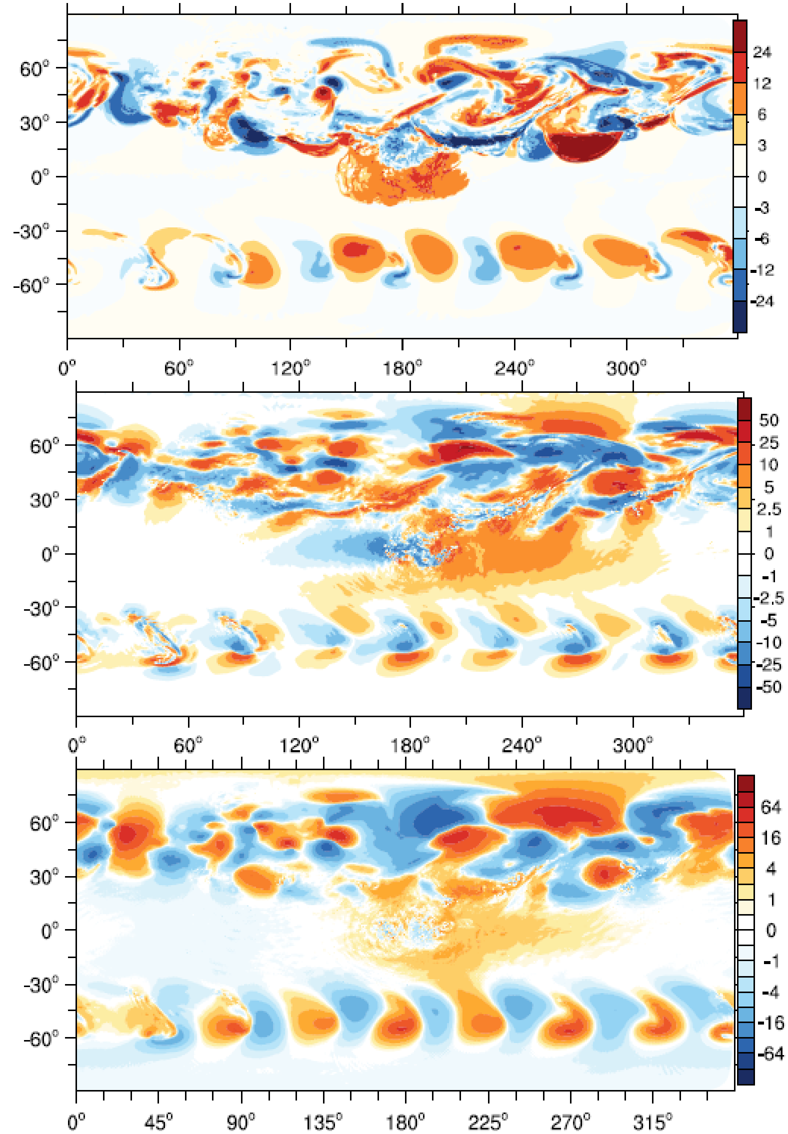


Fig. 7. The difference GBIS minus REF for surface potential temperature (top), zonal wind [m/s] at 850hPa (center) and surface pressure [hPa] (bottom) of the 18 days moist solutions.

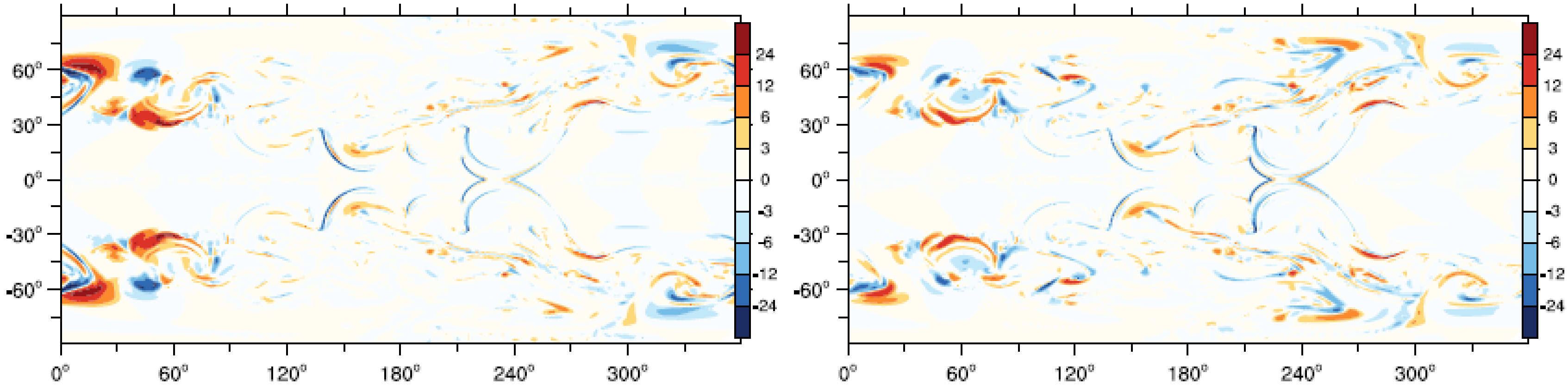


Fig. 5. GBIS-REF departures of the 18 days surface θ' solutions on the O180 grid for the symmetrically excited instability. Left and right panels show, respectively, the solutions with one (default) and two outer iterations in (16b).

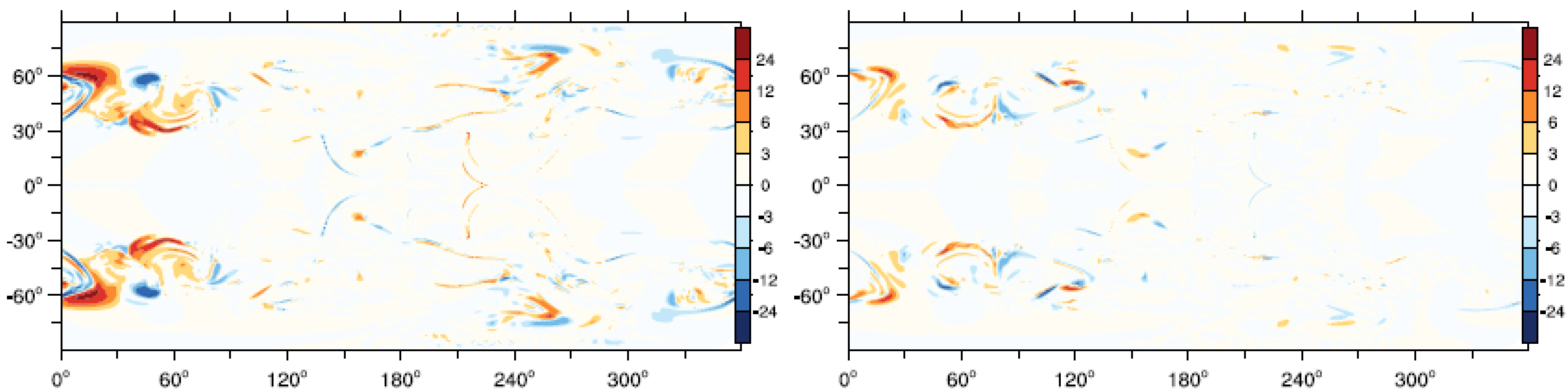


Fig. 6. The respective departures of REF (left) and GBIS (right) solutions with two and one (default) outer iterations.

The perturbation formulation offers: increased accuracy due to extending the trapezoidal rule on the convective derivative of the ambient velocity and generalised vectorial buoyancy; efficiency gains by preconditioning the solution procedure with arbitrary balanced states; and facilitated generalisations of geometry and physics.

We exploited a pristine approach with perturbations defined about the solutions of the generic PDEs. However, the resulting apparatus is equally applicable to approximate ambient states constructed based on alternative or surrogate models and/or data. The latter may require inclusion of additional forcing terms, but will not affect the machinery of the semi-implicit integrators and the BVP coefficients. Consequently, the new solvers enable the development of new multilevel methods, time parallel and/or hardware-failure resilient algorithms as well as blending with novel data-informed approaches.

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EXTRAS

Step 3: formulating linear problem

helpful notations

$$\theta/\theta_0 = \Theta$$
$$\mathbf{u} = (u^x, u^y, u^z)$$
$$\widetilde{\mathbf{G}}\nabla = \widetilde{\nabla} = (\widetilde{\partial_x}, \widetilde{\partial_y}, \widetilde{\partial_z})^{1} \quad \Rightarrow \quad \forall\ \psi,\quad (\widetilde{\mathbf{G}}^T\mathbf{u})\cdot\nabla\psi = \mathbf{u}\cdot(\widetilde{\mathbf{G}}\nabla\psi) \equiv \mathbf{u}\cdot\widetilde{\nabla}\psi$$

$$\tau^\theta := \delta_h t\,(1+\delta_h t\,\alpha^\theta)^{-1}\ ,\quad \tau^u := \delta_h t\,(1+\delta_h t\,\alpha^u)^{-1} \quad \Rightarrow$$

$$\theta'^{\,n+1}_{\mathrm{i}} = \widehat{\theta'}_{\mathrm{i}} - 0.5\delta t\left(\widetilde{\mathbf{G}}^T\mathbf{u}'^{\,n+1}\cdot\nabla\theta_a^{n+1} + \alpha^\theta\theta'^{\,n+1}\right)_{\mathrm{i}} \quad \Rightarrow \quad \Theta' = \frac{\tau^\theta}{\delta_h t}\left(\widehat{\Theta'} - \delta_h t\,\mathbf{u}'\,\widetilde{\nabla}\Theta_a\right)$$

thus leading to the shorthand of the momentum integral

$$\mathbf{u}' + \tau^u\mathbf{u}'\cdot\widetilde{\nabla}\mathbf{u}_a + \tau^u\mathbf{f}\times\mathbf{u}' - \tau^u\tau^\theta\left(\mathbf{u}'\cdot\widetilde{\nabla}\Theta_a\right)\widetilde{\nabla}\phi_a$$
$$= \widehat{\widehat{\mathbf{u}'}} - \tau^u\Theta^\star\,\widetilde{\nabla}\phi'\ .$$

where $$\widehat{\widehat{\mathbf{u}'}} = \frac{\tau^u}{\delta_h t}\left(\widehat{\mathbf{u}'} - \tau^\theta\widehat{\Theta'}\,\widetilde{\nabla}\phi_a + \delta_h t\,\mathcal{M}'(\mathbf{u}^\star, \mathbf{u}_a)\right)$$

$$\Rightarrow \qquad \mathbf{L}\,\mathbf{u}' = \widehat{\widehat{\mathbf{u}'}} - \tau^u\Theta^\star\,\widetilde{\nabla}\phi' \implies$$

$$\mathbf{u}' = \check{\check{\mathbf{u}}}' - \mathbf{C}\,\nabla\phi'\ ;\quad \check{\check{\mathbf{u}}}' = \mathbf{L}^{-1}\,\widehat{\widehat{\mathbf{u}'}}\ ,\quad \mathbf{C} = \tau^u\Theta^\star\,\mathbf{L}^{-1}\widetilde{\mathbf{G}}\ ,$$

[1] Consider that each $\widetilde{\partial}$ is composed of three terms; e.g., $\widetilde{\partial_x} = g_{11}\partial_x + g_{12}\partial_y + g_{13}\partial_z$, where g_{ij} are entires of $\widetilde{\mathbf{G}}$.

$$\mathbf{L}\,\mathbf{u}' = \widehat{\widehat{\mathbf{u}'}} - \tau^u \Theta^\star \,\widetilde{\nabla}\phi'$$ *in component notation* ➔

$$u'^{\,x} + \tau^u \left(u'^{\,x} \widetilde{\partial_x} u_a^x + u'^{\,y} \widetilde{\partial_y} u_a^x + u'^{\,z} \widetilde{\partial_z} u_a^x \right) + \tau^u \left(-u'^{\,y} f^z + u'^{\,z} f^y \right)$$
$$-\tau^u \tau^\theta \left(u'^{\,x} \widetilde{\partial_x} \Theta_a + u'^{\,y} \widetilde{\partial_y} \Theta_a + u'^{\,z} \widetilde{\partial_z} \Theta_a \right) \widetilde{\partial_x} \phi_a = \widehat{\widehat{u'^{\,x}}} - \tau^u \Theta^\star \widetilde{\partial_x} \phi' \;,$$

$$u'^{\,y} + \tau^u \left(u'^{\,x} \widetilde{\partial_x} u_a^y + u'^{\,y} \widetilde{\partial_y} u_a^y + u'^{\,z} \widetilde{\partial_z} u_a^y \right) + \tau^u \left(-u'^{\,z} f^x + u'^{\,x} f^z \right)$$
$$-\tau^u \tau^\theta \left(u'^{\,x} \widetilde{\partial_x} \Theta_a + u'^{\,y} \widetilde{\partial_y} \Theta_a + u'^{\,z} \widetilde{\partial_z} \Theta_a \right) \widetilde{\partial_y} \phi_a = \widehat{\widehat{u'^{\,y}}} - \tau^u \, \Theta^\star \widetilde{\partial_y} \phi' \;,$$

$$u'^{\,z} + \tau^u \left(u'^{\,x} \widetilde{\partial_x} u_a^z + u'^{\,y} \widetilde{\partial_y} u_a^z + u'^{\,z} \widetilde{\partial_z} u_a^z \right) + \tau^u \left(-u'^{\,x} f^y + u'^{\,y} f^x \right)$$
$$-\tau^u \tau^\theta \left(u'^{\,x} \widetilde{\partial_x} \Theta_a + u'^{\,y} \widetilde{\partial_y} \Theta_a + u'^{\,z} \widetilde{\partial_z} \Theta_a \right) \widetilde{\partial_z} \phi_a = \widehat{\widehat{u'^{\,z}}} - \tau^u \, \Theta^\star \widetilde{\partial_z} \phi' \;,$$

Which upon regrouping all the terms in the spirit of matrix-vector product ➔

$$u'^x[1+\tau^u(\widetilde{\partial_x u_a^x} - \tau^\theta \widetilde{\partial_x \Theta_a} \widetilde{\partial_x \phi_a})]$$
$$+u'^y[\tau^u(\widetilde{\partial_y u_a^x} - \tau^\theta \widetilde{\partial_y \Theta_a} \widetilde{\partial_x \phi_a} - f^z)]$$
$$+u'^z[\tau^u(\widetilde{\partial_z u_a^x} - \tau^\theta \widetilde{\partial_z \Theta_a} \widetilde{\partial_x \phi_a} + f^y)] = \widehat{\widehat{u'^x}} - \tau^u\, \Theta^\star \widetilde{\partial_x \phi'}$$

$$u'^x[\tau^u(\widetilde{\partial_x u_a^y} - \tau^\theta \widetilde{\partial_x \Theta_a} \widetilde{\partial_y \phi_a} + f^z)]$$
$$+u'^y[1+\tau^u(\widetilde{\partial_y u_a^y} - \tau^\theta \widetilde{\partial_y \Theta_a} \widetilde{\partial_y \phi_a})]$$
$$+u'^z[\tau^u(\widetilde{\partial_z u_a^y} - \tau^\theta \widetilde{\partial_z \Theta_a} \widetilde{\partial_y \phi_a} - f^x)] = \widehat{\widehat{u'^y}} - \tau^u\, \Theta^\star \widetilde{\partial_y \phi'}$$

$$u'^x[\tau^u(\widetilde{\partial_x u_a^z} - \tau^\theta \widetilde{\partial_x \Theta_a} \widetilde{\partial_z \phi_a} - f^y)]$$
$$+u'^y[\tau^u(\widetilde{\partial_y u_a^z} - \tau^\theta \widetilde{\partial_y \Theta_a} \widetilde{\partial_z \phi_a} + f^x)]$$
$$+u'^z[1+\tau^u(\widetilde{\partial_z u_a^z} - \tau^\theta \widetilde{\partial_z \Theta_a} \widetilde{\partial_z \phi_a})] = \widehat{\widehat{u'^z}} - \tau^u\, \Theta^\star \widetilde{\partial_z \phi'}$$

reveals all entries of the linear operator **L** ➔

$$l_{11} = 1 + \tau^u(\widetilde{\partial_x u_a^x} - \tau^\theta \widetilde{\partial_x \Theta_a} \widetilde{\partial_x \phi_a}) \,,$$
$$l_{12} = \tau^u(\widetilde{\partial_y u_a^x} - \tau^\theta \widetilde{\partial_y \Theta_a} \widetilde{\partial_x \phi_a} - f^z) \,,$$
$$l_{13} = \tau^u(\widetilde{\partial_z u_a^x} - \tau^\theta \widetilde{\partial_z \Theta_a} \widetilde{\partial_x \phi_a} + f^y) \,,$$

$$l_{21} = \tau^u(\widetilde{\partial_x u_a^y} - \tau^\theta \widetilde{\partial_x \Theta_a} \widetilde{\partial_y \phi_a} + f^z) \,,$$
$$l_{22} = 1 + \tau^u(\widetilde{\partial_y u_a^y} - \tau^\theta \widetilde{\partial_y \Theta_a} \widetilde{\partial_y \phi_a}) \,,$$
$$l_{23} = \tau^u(\widetilde{\partial_z u_a^y} - \tau^\theta \widetilde{\partial_z \Theta_a} \widetilde{\partial_y \phi_a} - f^x) \,,$$
$$l_{31} = \tau^u(\widetilde{\partial_x u_a^z} - \tau^\theta \widetilde{\partial_x \Theta_a} \widetilde{\partial_z \phi_a} - f^y) \,,$$
$$l_{32} = \tau^u(\widetilde{\partial_y u_a^z} - \tau^\theta \widetilde{\partial_y \Theta_a} \widetilde{\partial_z \phi_a}) + f^x) \,,$$
$$l_{33} = 1 + \tau^u(\widetilde{\partial_z u_a^z} - \tau^\theta \widetilde{\partial_z \Theta_a} \widetilde{\partial_z \phi_a}) \,.$$

$$\mathscr{E}^{-1} = [\mathscr{G}_3\widetilde{\mathscr{F}}_2\mathscr{F}_3 + \mathscr{G}_2(1+\widetilde{\mathscr{F}}_2\mathscr{F}_2)]\vartheta_y + [\mathscr{G}_2\mathscr{F}_2\widetilde{\mathscr{F}}_3 + \mathscr{G}_3(1 + \widetilde{\mathscr{F}}_3\mathscr{F}_3)]\vartheta_z$$
$$+(1+\tilde{\alpha}^*)(1 + \widetilde{\mathscr{F}}_2\mathscr{F}_2 + \widetilde{\mathscr{F}}_3\mathscr{F}_3) \ .$$

$$\widetilde{\mathscr{F}}_2 = \mathscr{F}_2 + \widetilde{\partial}_z^* u_a \ , \quad \widetilde{\mathscr{F}}_3 = \mathscr{F}_3 - \widetilde{\partial}_y^* u_a,$$

$$\mathscr{V}^1 = \mathscr{A}U + \mathscr{B}V - \mathscr{X}W \ ,$$
$$\mathscr{V}^2 = \mathscr{C}U + \mathscr{D}V + \mathscr{Y}W \ ,$$
$$\mathscr{V}^3 = \mathscr{H}U + \mathscr{I}V + \mathscr{Z}W \ .$$

$$\mathscr{A} = \ \mathscr{R} + \mathscr{G}_2\vartheta_y + \mathscr{G}_3\vartheta_z \ ,$$
$$\mathscr{B} = \ \mathscr{R}\widetilde{\mathscr{F}}_3 + \mathscr{G}_3(\widetilde{\mathscr{F}}_2\vartheta_y + \widetilde{\mathscr{F}}_3\vartheta_z) \ ,$$
$$\mathscr{X} = -\mathscr{R}\widetilde{\mathscr{F}}_2 - \mathscr{G}_2(\widetilde{\mathscr{F}}_2\vartheta_y + \widetilde{\mathscr{F}}_3\vartheta_z) \ ,$$

$$\mathscr{C}^{11} = \mathscr{R}(\widetilde{G}_1^1 + \widetilde{G}_2^1\widetilde{\mathscr{F}}_3) + \mathscr{G}_2\widetilde{G}_1^1\vartheta_y + \mathscr{G}_3\left[\widetilde{G}_2^1\widetilde{\mathscr{F}}_2\vartheta_y + (\widetilde{G}_1^1 + \widetilde{G}_2^1\widetilde{\mathscr{F}}_3)\vartheta_z\right]$$
$$\mathscr{C}^{12} = \mathscr{R}(\widetilde{G}_1^2 + \widetilde{G}_2^2\widetilde{\mathscr{F}}_3) + \mathscr{G}_2\widetilde{G}_1^2\vartheta_y + \mathscr{G}_3\left[\widetilde{G}_2^2\widetilde{\mathscr{F}}_2\vartheta_y + (\widetilde{G}_1^2 + \widetilde{G}_2^2\widetilde{\mathscr{F}}_3)\vartheta_z\right]$$
$$\mathscr{C}^{13} = \mathscr{R}(\widetilde{G}_1^3 + \widetilde{G}_2^3\widetilde{\mathscr{F}}_3 - \widetilde{G}_3^3\widetilde{\mathscr{F}}_2) + \mathscr{G}_3\left[\ \widetilde{G}_2^3\widetilde{\mathscr{F}}_2\vartheta_y + (\widetilde{G}_1^3 + \widetilde{G}_2^3\widetilde{\mathscr{F}}_3)\vartheta_z\right]$$
$$+ \mathscr{G}_y\left[-\widetilde{G}_3^3\widetilde{\mathscr{F}}_3\vartheta_z + (\widetilde{G}_1^3 - \widetilde{G}_3^3\widetilde{\mathscr{F}}_2)\vartheta_y\right]$$