1. Introduction

This technical report is a description of the fifth-generation Penn State/NCAR Mesoscale Model (MM5). It is based on the original version described by Anthes and Warner (1978). Although a few of the following details of this model are well represented in Anthes *et al.* (1987), extensive changes and increases in options have occurred. For completeness, those parts that have changed little or none will also be represented here. The document structure is as follows. In section 2 we will describe the governing equations, algorithms, and boundary conditions. This will include the finite difference algorithms and time splitting techniques of both the hydrostatic and the nonhydrostatic equations of motion (hydrostatic and nonhydrostatic solver). All subsequent sections will describe features common to both solvers. Section 3 will discuss the mesh-refinement scheme, section 4 the four-dimensional data-assimilation technique, and section 5 will focus on the various physics options.

2. Governing equations and numerical algorithms

2.1 Hydrostatic model equations

The vertical σ-coordinate is defined in terms of pressure.

$$\sigma = \frac{p - p_t}{p_s - p_t},$$

where p_s and p_t are the surface and top pressures respectively of the model, where p_t is a constant.

The model equations are given by the following, where $p^* = p_s - p_t$.

Horizontal momentum;

$$\frac{\partial p^* u}{\partial t} = -m^2 \left[\frac{\partial p^* uu/m}{\partial x} + \frac{\partial p^* vu/m}{\partial y}\right] - \frac{\partial p^* u\dot{\sigma}}{\partial \sigma} - mp^* \left[\frac{\sigma}{\rho}\frac{\partial p^*}{\partial x} + \frac{\partial \phi}{\partial x}\right] + p^* fv + D_u \tag{2.1.1}$$

$$\frac{\partial p^* v}{\partial t} = -m^2 \left[\frac{\partial p^* uv/m}{\partial x} + \frac{\partial p^* vv/m}{\partial y}\right] - \frac{\partial p^* v\dot{\sigma}}{\partial \sigma} - mp^* \left[\frac{\sigma}{\rho}\frac{\partial p^*}{\partial y} + \frac{\partial \phi}{\partial y}\right] - p^* fu + D_v \tag{2.1.2}$$

Temperature;

$$\frac{\partial p^* T}{\partial t} = - m^2 \left[\frac{\partial p^* u T/m}{\partial x} + \frac{\partial p^* v T/m}{\partial y} \right] - \frac{\partial p^* T \dot{\sigma}}{\partial \sigma}$$

$$+ p^* \frac{\omega}{\rho c_p} + p^* \frac{\dot{Q}}{c_p} + D_T, \tag{2.1.3}$$

where the D terms represent the vertical and horizontal diffusion terms and vertical mixing due to the planetary boundary layer turbulence or dry convective adjustment. The heat capacity for moist air at constant pressure is given by $c_p = c_{pd}(1 + 0.8 q_v)$, where q_v is the mixing ratio for water vapor and c_{pd} is the heat capacity for dry air.

Surface pressure is computed from

$$\frac{\partial p^*}{\partial t} = -m^2 \left[\frac{\partial p^* u/m}{\partial x} + \frac{\partial p^* v/m}{\partial y} \right] - \frac{\partial p^* \dot{\sigma}}{\partial \sigma}, \tag{2.1.4}$$

which is used in its vertically integrated form

$$\frac{\partial p^*}{\partial t} = -m^2 \int_0^1 \left[\frac{\partial p^* u/m}{\partial x} + \frac{\partial p^* v/m}{\partial y} \right] d\sigma. \tag{2.1.5}$$

Then the vertical velocity in σ-coordinates, $\dot{\sigma}$, is computed from (2.1.4) by vertical integration. Thus

$$\dot{\sigma} = -\frac{1}{p^*} \int_0^\sigma \left[\frac{\partial p^*}{\partial t} + m^2 \left(\frac{\partial p^* u/m}{\partial x} + \frac{\partial p^* v/m}{\partial y} \right) \right] d\sigma', \tag{2.1.6}$$

where σ' is a dummy variable of integration and $\dot{\sigma}(\sigma = 0) = 0$.

In the thermodynamic equation, (2.1.3), $\omega = \frac{dp}{dt}$ and is calculated from

$$\omega = p^* \dot{\sigma} + \sigma \frac{dp^*}{dt}, \tag{2.1.7}$$

where

$$\frac{dp^*}{dt} = \frac{\partial p^*}{\partial t} + m \left[u \frac{\partial p^*}{\partial x} + v \frac{\partial p^*}{\partial y} \right]. \tag{2.1.8}$$

The hydrostatic equation is used to compute the geopotential heights from the virtual temperature, T_v:

$$\frac{\partial \phi}{\partial ln(\sigma + p_t/p^*)} = -RT_v \left[1 + \frac{q_c + q_r}{1 + q_v} \right]^{-1}, \tag{2.1.9}$$

where T_v is given by $T_v = T(1 + 0.608q_v)$, and q_c and q_r are the mixing ratios of cloud water and rain water.

2.2 Nonhydrostatic model equations

For the nonhydrostatic model we define a constant reference state and perturbations from it, as follows:

$$\begin{aligned} p(x,y,z,t) &= p_0(z) + p'(x,y,z,t), \\ T(x,y,z,t) &= T_0(z) + T'(x,y,z,t), \\ \rho(x,y,z,t) &= \rho_0(z) + \rho'(x,y,z,t). \end{aligned}$$

Typically the temperature profile for the reference state may be an analytic function that fits the mean tropospheric temperature profile.

The vertical σ-coordinate is then defined entirely from the reference pressure.

$$\sigma = \frac{p_0 - p_t}{p_s - p_t},$$

where p_s and p_t are the surface and top pressures respectively of the reference state and are independent of time. The total pressure at a grid point is therefore given by

$$p = p^*\sigma + p_t + p',$$

where $p^*(x,y) = p_s(x,y) - p_t$. The three-dimensional pressure perturbation, p', is a predicted quantity.

The model equations (Dudhia 1993) are then given by the following:

Horizontal momentum;

$$\begin{aligned} \frac{\partial p^* u}{\partial t} &= -m^2\left[\frac{\partial p^* uu/m}{\partial x} + \frac{\partial p^* vu/m}{\partial y}\right] - \frac{\partial p^* u\dot\sigma}{\partial \sigma} + uDIV \\ &\quad - \frac{mp^*}{\rho}\left[\frac{\partial p'}{\partial x} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial x}\frac{\partial p'}{\partial \sigma}\right] + p^* fv + D_u \qquad (2.2.1) \\ \frac{\partial p^* v}{\partial t} &= -m^2\left[\frac{\partial p^* uv/m}{\partial x} + \frac{\partial p^* vv/m}{\partial y}\right] - \frac{\partial p^* v\dot\sigma}{\partial \sigma} + vDIV \\ &\quad - \frac{mp^*}{\rho}\left[\frac{\partial p'}{\partial y} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial y}\frac{\partial p'}{\partial \sigma}\right] - p^* fu + D_v \qquad (2.2.2) \end{aligned}$$

Vertical momentum;

$$\frac{\partial p^* w}{\partial t} = -m^2 \left[\frac{\partial p^* uw/m}{\partial x} + \frac{\partial p^* vw/m}{\partial y} \right] - \frac{\partial p^* w \dot{\sigma}}{\partial \sigma} + wDIV$$

$$+ p^* g \frac{\rho_0}{\rho} \left[\frac{1}{p^*} \frac{\partial p'}{\partial \sigma} + \frac{T_v'}{T} - \frac{T_0 p'}{T p_0} \right] - p^* g \left[(q_c + q_r) \right] + D_w \qquad (2.2.3)$$

Pressure;

$$\frac{\partial p^* p'}{\partial t} = -m^2 \left[\frac{\partial p^* up'/m}{\partial x} + \frac{\partial p^* vp'/m}{\partial y} \right] - \frac{\partial p^* p' \dot{\sigma}}{\partial \sigma} + p' DIV$$

$$- m^2 p^* \gamma p \left[\frac{\partial u/m}{\partial x} - \frac{\sigma}{mp^*} \frac{\partial p^*}{\partial x} \frac{\partial u}{\partial \sigma} + \frac{\partial v/m}{\partial y} - \frac{\sigma}{mp^*} \frac{\partial p^*}{\partial y} \frac{\partial v}{\partial \sigma} \right]$$

$$+ \rho_0 g \gamma p \frac{\partial w}{\partial \sigma} + p^* \rho_0 g w \qquad (2.2.4)$$

Temperature;

$$\frac{\partial p^* T}{\partial t} = -m^2 \left[\frac{\partial p^* uT/m}{\partial x} + \frac{\partial p^* vT/m}{\partial y} \right] - \frac{\partial p^* T \dot{\sigma}}{\partial \sigma} + T\, DIV$$

$$+ \frac{1}{\rho c_p} \left[p^* \frac{Dp'}{Dt} - \rho_0 g p^* w - D_{p'} \right] + p^* \frac{\dot{Q}}{c_p} + D_T, \qquad (2.2.5)$$

where

$$DIV = m^2 \left[\frac{\partial p^* u/m}{\partial x} + \frac{\partial p^* v/m}{\partial y} \right] + \frac{\partial p^* \dot{\sigma}}{\partial \sigma}, \qquad (2.2.6)$$

and

$$\dot{\sigma} = -\frac{\rho_0 g}{p^*} w - \frac{m\sigma}{p^*} \frac{\partial p^*}{\partial x} u - \frac{m\sigma}{p^*} \frac{\partial p^*}{\partial y} v. \qquad (2.2.7)$$

The DIV terms are not in the hydrostatic equations and arise because p^* is now constant in time. Thus the hydrostatic continuity equation no longer applies, leaving the right hand side terms in (2.2.6) uncancelled by the surface pressure tendency. The equations are thus in advective form.

Equation (2.2.4) can be derived from the fully compressible mass continuity relation and the perfect gas law. The only term neglected in equations (2.2.1)-(2.2.5) is a diabatic term contributing to the perturbation pressure tendency in (2.2.4). This term is negligible in normal meteorological regimes since it only forces a small divergence (i.e. expansion) in regions of heating. The $D_{p'}$ term in (2.2.5) is a small correction to D_T allowing for horizontal pressure differences in thermal diffusion.

2.2.1 Complete Coriolis force option

In the nonhydrostatic model it is possible to include the other components of the Coriolis force that are neglected in the traditional approximation. The full Coriolis force leads to a small upward/downward acceleration on westerly/easterly flows and an westward/eastward acceleration on upward/downward flows in addition to the rightward/leftward deflection of horizontal flows in the northern/southern hemisphere.

To determine this force, two additional parameters are defined. We will refer to the other component of the Coriolis parameter as $e = 2\Omega\cos\phi$, where Ω is the angular velocity of the earth and ϕ is the latitude. The other new parameter is θ, which is the angular difference between the y-axis of the grid and true north. It is found from

$$\tan\theta = -\cos\phi\frac{\partial\lambda/\partial y}{\partial\phi/\partial y}, \tag{2.2.8}$$

where λ is longitude. A special provision is made for the dateline. Thus θ is positive if north is rotated clockwise from the y-axis.

The momentum equations are then given by the following:

Horizontal momentum

$$\begin{aligned}\frac{\partial p^*u}{\partial t} &= -m^2\left[\frac{\partial p^*uu/m}{\partial x} + \frac{\partial p^*vu/m}{\partial y}\right] - \frac{\partial p^*u\dot\sigma}{\partial\sigma} + uDIV \\ &- \frac{mp^*}{\rho}\left[\frac{\partial p'}{\partial x} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial x}\frac{\partial p'}{\partial\sigma}\right] + p^*fv - p^*ew\cos\theta + D_u \end{aligned}\tag{2.2.9}$$

$$\begin{aligned}\frac{\partial p^*v}{\partial t} &= -m^2\left[\frac{\partial p^*uv/m}{\partial x} + \frac{\partial p^*vv/m}{\partial y}\right] - \frac{\partial p^*v\dot\sigma}{\partial\sigma} + vDIV \\ &- \frac{mp^*}{\rho}\left[\frac{\partial p'}{\partial y} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial y}\frac{\partial p'}{\partial\sigma}\right] - p^*fu + p^*ew\sin\theta + D_v \end{aligned}\tag{2.2.10}$$

Vertical momentum;

$$\begin{aligned}\frac{\partial p^*w}{\partial t} &= -m^2\left[\frac{\partial p^*uw/m}{\partial x} + \frac{\partial p^*vw/m}{\partial y}\right] - \frac{\partial p^*w\dot\sigma}{\partial\sigma} + wDIV \\ &+ p^*g\frac{\rho_0}{\rho}\left[\frac{1}{p^*}\frac{\partial p'}{\partial\sigma} + \frac{T_v'}{T} - \frac{T_0p'}{Tp_0}\right] - p^*g\left[(q_c + q_r)\right] \\ &+ p^*e(u\cos\theta - v\sin\theta) + D_w. \end{aligned}\tag{2.2.11}$$

2.3 Nonhydrostatic Finite Difference Algorithms

The B-grid staggering of horizontal velocity variables with respect to the other fields is shown in Fig. 2.1. Vertical velocity is staggered vertically. Noting that the j index increments in the x direction, and i in the y direction, the conventional notation will be as follows.

$$a_x \;=\; (a_{i,j+\frac{1}{2}} \;-\; a_{i,j-\frac{1}{2}})/\Delta x. \tag{2.3.1}$$

$$\overline{a}^x \;=\; \frac{1}{2}(a_{i,j+\frac{1}{2}} \;+\; a_{i,j-\frac{1}{2}}), \tag{2.3.2a}$$

Multiple averaging terms such as $\overline{a}^{xyy}$ can also be defined as successive averages where the order of superscripts does not matter, e.g.,

$$\overline{a}^{xyy} \;=\; \overline{\overline{\overline{a}^{x}}^{y}}^{y}.$$

Averaging vertically allows for non-uniform grid-lengths and nonlinearly varying fields, such as temperature and water vapor, by suitably weighting the values.

Thus for half-level fields averaged to full levels

$$\overline{a}^{\sigma} \;=\; \frac{a_{k+\frac{1}{2}}(\sigma_k - \sigma_{k-\frac{1}{2}}) \;+\; a_{k-\frac{1}{2}}(\sigma_{k+\frac{1}{2}} - \sigma_k)}{(\sigma_{k+\frac{1}{2}} - \sigma_{k-\frac{1}{2}})}, \tag{2.3.2b}$$

while averaging full-level fields to half levels uses an equation similar to (2.3.2a). For temperature, a is the potential temperature, and for water vapor, a is log q_v.

The spatial differencing of the terms in the horizontal momentum prediction equations is [including the map-scale factor $m(x,y)$],

$$\begin{aligned}\frac{\partial p_d^* u}{\partial t} \;=\; & -\, m^2 \left[\left(\overline{u}^x \, \overline{\frac{p_d^* u}{m}}^{xyy} \right)_x \;+\; \left(\overline{u}^y \, \overline{\frac{p_d^* v}{m}}^{xyx} \right)_y \right] \;-\; \left(\overline{p_d^* u}^{\sigma} \, \overline{\dot{\sigma}}^{xy} \right)_{\sigma} \\ & +\, u\overline{DIV}^{xy} \;-\; \frac{m p_d^*}{\overline{\rho}^{xy}} \left[\overline{p'_x}^{y} \;-\; \overline{(\sigma p^*)_x}^{y} \, \overline{\frac{p'_\sigma}{p^*}}^{xy\sigma} \right] \\ & +\, p_d^* f v \;+\; D(p_d^* u),\end{aligned} \tag{2.3.3}$$

$$\frac{\partial p_d^* v}{\partial t} = -m^2\left[\left(\overline{v}^x \overline{\frac{p_d^* u}{m}}^{xyy}\right)_x + \left(\overline{v}^y \overline{\frac{p_d^* v}{m}}^{xyx}\right)_y\right] - (\overline{p_d^* v}^\sigma \overline{\dot\sigma}^{xy})_\sigma$$

$$+ v\overline{DIV}^{xy} - \frac{m p_d^*}{\overline{\rho}^{xy}}\left[\overline{p'_y}^x - \overline{(\sigma p^*)_y}^x \overline{\frac{p'_\sigma}{p^*}}^{xy\sigma}\right]$$

$$- p_d^* f u + D(p_d^* v), \qquad (2.3.4)$$

where $p_d^* = \overline{p^*}^{xy}$, and DIV, the mass divergence term, is given by

$$DIV = m^2\left[\left(\overline{\frac{p_d^* u}{m}}^y\right)_x + \left(\overline{\frac{p_d^* v}{m}}^x\right)_y\right] + p^*\dot\sigma_\sigma. \qquad (2.3.5)$$

The triple averaging in the horizontal momentum advection terms follows that of the hydrostatic model as discussed by Anthes (1972). The subgrid-scale and diffusion operators are represented by $D(a) = K_h \Delta x^2 (a_{xxxx} + a_{yyyy}) + (K_v a_z)_z +$ (PBL tendencies), where the fourth-order scheme is modified to second-order near the boundaries.

The coordinate vertical velocity, $\dot\sigma$, is obtained from

$$\dot\sigma = -\frac{\overline{\rho_0}^\sigma g}{p^*} w - \frac{m\sigma}{p^*}\overline{p^*_x}^x \overline{u}^{xy\sigma} - \frac{m\sigma}{p^*}\overline{p^*_y}^y \overline{v}^{xy\sigma}, \qquad (2.3.6)$$

and the vertical momentum equation is

$$\frac{\partial p^* w}{\partial t} = -m^2\left[\left(\overline{w}^x \overline{\frac{p^* u}{m}}^{y\sigma}\right)_x + \left(\overline{w}^y \overline{\frac{p^* v}{m}}^{x\sigma}\right)_y\right] - (\overline{p^* w}^\sigma \overline{\dot\sigma}^\sigma)_\sigma$$

$$+ w\overline{DIV}^\sigma + p^* g \overline{\frac{\rho_0}{\rho}}^\sigma \left[\frac{1}{p^*} p'_\sigma - \frac{1}{\gamma}\overline{\frac{p' T_0}{p_0 T}}^\sigma\right]$$

$$+ p^* g \overline{\frac{\rho_0}{\rho}}^\sigma \left[\overline{\frac{T'_v}{T}}^\sigma - \frac{R}{c_p}\overline{\frac{p' T_0}{p_0 T}}^\sigma\right] - p^* g \overline{(q_c + q_r)}^\sigma + D(p^* w). \qquad (2.3.7)$$

The pressure tendency equation, neglecting diabatic terms, is given by

$$\frac{\partial p^* p'}{\partial t} = -m^2\left[\left(\overline{p'}^x \overline{\frac{p^* u}{m}}^y\right)_x + \left(\overline{p'}^y \overline{\frac{p^* v}{m}}^x\right)_y\right] - (\overline{p^* p'}^\sigma \dot\sigma)_\sigma$$

$$+ p' DIV + p^* \rho_0 g \overline{w}^\sigma - m^2 p^* \gamma p\left[\left(\overline{\frac{u}{m}}^y\right)_x - (\sigma\overline{p^*}^x)_x \frac{1}{m p^*}\overline{u_\sigma}^{xy\sigma}\right.$$

$$+\left(\frac{\overline{v}^{x}}{m}\right)_{y} - (\sigma\overline{p^*}^{y})_y\frac{1}{mp^*}\overline{v_\sigma}^{xy\sigma} - \frac{\rho_0 g}{m^2p^*}w_\sigma\Bigg], \qquad (2.3.8)$$

and temperature tendency is differenced as

$$\frac{\partial p^*T}{\partial t} = -m^2\left[\left(\overline{T}^{x}\,\overline{\frac{p^*u}{m}}^{y}\right)_x + \left(\overline{T}^{y}\,\overline{\frac{p^*v}{m}}^{x}\right)_y\right] - (\overline{p^*T}^{\sigma}\dot{\sigma})_\sigma$$

$$+ T\,DIV + \frac{1}{\rho c_p}\left[p^*\frac{Dp'}{Dt} - \rho_0 g\overline{p^*w}^{\sigma} - D(p^*p')\right]$$

$$+ p^*\frac{\dot{Q}}{c_p} + D(p^*T), \qquad (2.3.9)$$

where Dp'/Dt is differenced like the corresponding terms in (2.3.8). Moisture variables have similar advection forms to those in (2.3.8) and (2.3.9) except when using the upstream option where $\overline{q}^{x}$ is replaced by the upstream value alone.

2.4 Hydrostatic Finite Difference Algorithms

The hydrostatic finite differencing of advection, Coriolis and heating follows (2.3.3), (2.3.4) and (2.3.9) without the DIV terms. The pressure gradient terms in (2.3.3) become

$$PG \;=\; -mp_d^*\overline{\phi_x}^y \;-\; \frac{mR\overline{T_v}^{xy}}{(1 \;+\; p_t/p_d^*)}\overline{p_x^*}^y, \qquad (2.4.1)$$

and likewise for the y-gradient in (2.3.4). The surface pressure tendency is found from the integration over all $(KMAX)$ layers of thickness $\delta\sigma(k)$,

$$\frac{\partial p^*}{\partial t} \;=\; -m^2 \sum_{k=1}^{KMAX} \left[\left(\frac{\overline{p_d^* u}^y}{m}\right)_x \;+\; \left(\frac{\overline{p_d^* v}^x}{m}\right)_y \right] \delta\sigma(k). \qquad (2.4.2)$$

Then $\dot{\sigma}$ is found from downward integration,

$$\dot{\sigma}(k+1) \;=\; \dot{\sigma}(k) \;-\; \frac{\partial p^*}{\partial t}\frac{\delta\sigma(k)}{p^*} \;-\; m^2 \left[\left(\frac{\overline{p_d^* u}^y}{m}\right)_x \;+\; \left(\frac{\overline{p_d^* v}^x}{m}\right)_y \right] \frac{\delta\sigma(k)}{p^*}, \qquad (2.4.3)$$

using the upper boundary condition that $\dot{\sigma}(k = 1) = 0$. The adiabatic term in (2.3.9), represented by the second set of terms in square brackets, becomes $p^*\omega$ in the hydrostatic model, where ω is defined by

$$\omega \;=\; \frac{dp}{dt} \;=\; p^*\overline{\dot{\sigma}}^\sigma \;+\; \sigma\left(\frac{\partial p^*}{\partial t} \;+\; m\overline{u}^{xy}\overline{p_x^*}^x \;+\; m\overline{v}^{xy}\overline{p_y^*}^y\right). \qquad (2.4.4)$$

The integration of the hydrostatic equation to obtain geopotential height, ϕ, in the hydrostatic model is done as follows.

$$\delta\phi \;=\; -\; R\overline{T_v L}^\sigma \delta ln(\sigma + p_t/p^*), \qquad (2.4.5)$$

where

$$L \;=\; \left[1 \;+\; \frac{q_c \;+\; q_r}{1 \;+\; q_v}\right]^{-1},$$

and allows for water loading when the explicit moisture scheme is used. Because ϕ is required on the velocity levels (half-levels), it has to be integrated first between the surface, where $\sigma = 1$ and $\phi = gh$ (h is the terrain height above sea-level), and the lowest half-level using (2.4.5) with just the lowest-level values T_v, q_v, q_c, q_r. At all other levels (2.4.5) uses vertical averaging between two levels.

The temporal differencing in the hydrostatic and nonhydrostatic models consists of leapfrog steps with an Asselin filter. With this time filter, splitting of the solution often associated with the leapfrog scheme is avoided. It is applied to all variables as

$$\hat{\alpha}^t = (1 - 2\nu)\alpha^t + \nu(\alpha^{t+1} + \hat{\alpha}^{t-1}), \tag{2.4.6}$$

where $\hat{\alpha}$ is the filtered variable. The coefficient ν in the model is 0.1 for all variables. For stability, diffusion terms are evaluated on the variables at time $t - 1$, as are the terms associated with the moist physical processes.

2.5 Time splitting

In both the nonhydrostatic as well as the hydrostatic numerics, a time splitting scheme is applied to increase efficiency. Because the nonhydrostatic equations above are fully compressible, they permit sound waves. These are fast and require a short time step for numerical stability. For the hydrostatic equations, fast moving external gravity waves are the limiting factor. The techniques described next are designed to split these fast moving waves from the rest of the solution.

2.5.1 The nonhydrostatic semi-implicit scheme

For the nonhydrostatic equations it is possible to separate terms directly involved with acoustic waves from comparatively slowly varying terms, and to handle the former with shorter time steps while updating the slow terms less frequently. The reduced equation set for the short time step makes the model more efficient. The separated equations only contain interactions between momentum and pressure and can be written as:

Horizontal momentum;

$$\frac{\partial u}{\partial t} + \frac{m}{\rho}\left[\frac{\partial p'}{\partial x} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial x}\frac{\partial p'}{\partial \sigma}\right] = S_u \tag{2.5.1.1}$$

$$\frac{\partial v}{\partial t} + \frac{m}{\rho}\left[\frac{\partial p'}{\partial y} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial y}\frac{\partial p'}{\partial \sigma}\right] = S_v \tag{2.5.1.2}$$

Vertical momentum;

$$\frac{\partial w}{\partial t} - \frac{\rho_0}{\rho}\frac{g}{p^*}\frac{\partial p'}{\partial \sigma} + \frac{g}{\gamma}\frac{p'}{p} = S_w \tag{2.5.1.3}$$

Pressure;

$$\frac{\partial p'}{\partial t} + m^2\gamma p\left[\frac{\partial u/m}{\partial x} - \frac{\sigma}{mp^*}\frac{\partial p^*}{\partial x}\frac{\partial u}{\partial \sigma} + \frac{\partial v/m}{\partial y} - \frac{\sigma}{mp^*}\frac{\partial p^*}{\partial y}\frac{\partial v}{\partial \sigma}\right]$$

$$-\frac{\rho_0 g \gamma p}{p^*}\frac{\partial w}{\partial \sigma} - \rho_0 g w = S_{p'}, \qquad (2.5.1.4)$$

where the S terms contain advection, diffusion, buoyancy and Coriolis tendencies. These are kept constant during the sub-steps. Note that only part of the p'/p term is in (2.5.1.3), where the rest has been absorbed in the buoyancy term that contributes to S_w.

The method of solution follows the semi-implicit scheme of Klemp and Wilhelmson (1978) for the short time step. Starting with u, v, w, p' known at time τ, first the two horizontal momentum equations are stepped forward to give $u^{\tau+1}$ and $v^{\tau+1}$ which are then used in the pressure equation, giving a time-centered explicit treatment of horizontally propagating sound waves. Vertical propagation of sound waves is treated implicitly by making $w^{\tau+1}$ and $p'^{\tau+1}$ depend upon time-averaged values of p' and w respectively in (2.5.1.3) and (2.5.1.4). For instance, where p' appears in (2.5.1.3) it is represented by

$$\overline{p'}^{\beta} = \frac{1}{2}(1 + \beta){p'}^{\tau+1} + \frac{1}{2}(1 - \beta){p'}^{\tau},$$

and similarly for w in (2.5.1.4). The parameter β determines the time-weighting, where zero gives a time-centered average and positive values give a bias towards the future time step that can be used for acoustic damping. In practice, values of $\beta = 0.2 - 0.4$ are used.

With second-order vertical spatial derivatives the finite difference forms of equations (2.5.1.3) and (2.5.1.4) can be combined, eliminating ${p'}^{\tau+1}$, into a finite difference equation for $w^{\tau+1}$, which is solvable by direct recursion on a tri-diagonal matrix.

The implicit vertical differencing scheme allows the short time step to be independent of the vertical resolution of the model, which is important for efficiency, and thus the step only depends upon the horizontal grid length. Additionally, the divergence damping technique of Skamarock and Klemp (1992) is used to control horizontally propagating sound waves. This method is similar to using time-extrapolated pressure terms in (2.5.1.1) and (2.5.1.2), where in practice the extrapolation is about 0.1 $\Delta\tau$.

Temperature and moisture are predicted using the normal leapfrog step, Δt, because they have no high-frequency terms contributing to acoustic waves. The slow terms for momentum and pressure contained in the S-terms above are also evaluated on these leapfrog steps, but for these variables the march from $t - \Delta t$ to $t + \Delta t$ is split into typically four steps of length $\Delta\tau$ during which momentum and pressure are continually updated.

2.5.2 The hydrostatic split-explicit scheme

When numerically solving the hydrostatic equations of motion, the stability criterion is severely limited by external gravity waves. These are very fast moving gravity waves that are small in amplitude (quasi-linear) and contain only a small fraction of the total energy. Hence they change slowly over the time scale of the Rossby waves. Because of this, splitting methods have been developed to split these fast waves from the solution (similar also to the above method for the nonhydrostatic equations to split sound-waves). From all the existing different options, we have chosen a method developed by Madala (1981). This scheme separates the terms governing the gravity modes from those governing the Rossby modes. The term "split" here refers to the separation of the motion in terms of eigenmodes. Similar to the nonhydrostatic method, the equations are rewritten in finite difference form as

$$\frac{\partial P_s u}{\partial t} + \delta_x \Phi = A_u, \tag{2.5.2.1}$$

$$\frac{\partial P_s v}{\partial t} + \delta_y \Phi = A_v, \tag{2.5.2.2}$$

$$\frac{\partial P_s T}{\partial t} + M_2 \cdot D = A_T, \tag{2.5.2.3}$$

$$\frac{\partial P_s}{\partial t} + N_1 \cdot D = 0, and \tag{2.5.2.4}$$

$$\Phi = M_1 \cdot T. \tag{2.5.2.5}$$

where the right hand sides change slowly over the time scale of the Rossby-waves. Matrices M_1, M_2, and vector N_1 are independent of x, y, and t. Notice the similarity to the nonhydrostatic splitting method (equations 2.5.1-2.5.4). However, rather then integrating the "fast" terms on a small time-step directly, the method described below only computes correction terms to the equations, making this process extremely efficient. To illustrate this, we follow Madala (1981). From the governing equations he derives equations for the mass divergence D and the generalized geopotential Φ. They are

$$\frac{\partial D}{\partial t} + [\delta_x^2 + \delta_y^2]\Phi = \delta_x A_u + \delta_y A_v \tag{2.5.2.6}$$

and

$$\frac{\partial \Phi}{\partial t} + M_3 \cdot D = M_1 \cdot A_T. \tag{2.5.2.7}$$

Integrating equations (2.5.2.1-2.5.2.3) from $t - \Delta t$ to $t + \Delta t$, where Δt is the time step of the slow Rossby modes, one gets

$$p_s u(t+\Delta t) - p_s u(t-\Delta t) + 2\Delta t \delta_x \tilde{\Phi} = 2\Delta t A_u(t), \tag{2.5.2.8}$$

$$p_s v(t+\Delta t) - p_s v(t-\Delta t) + 2\Delta t \delta_y \tilde{\Phi} = 2\Delta t A_v(t), \tag{2.5.2.9}$$

$$p_s T(t+\Delta t) - p_s T(t-\Delta t) + 2\Delta t M_2 \tilde{\Phi} = 2\Delta t A_T(t), \tag{2.5.2.10}$$

where the operator $\tilde{()}$ for the split-explicit scheme is defined as

$$\tilde{\beta} = \frac{\Delta\tau}{\Delta t}\sum_{n=1}^{m} \beta(t - \Delta t + n\Delta\tau),$$

where $m = \frac{\Delta\tau}{\Delta t}$. Denoting with superscript *ex* solutions computed using only the explicit time integration over $2\Delta t$, equations (2.5.2.8-2.5.2.10) can be written as

$$p_s u(t+\Delta t) + 2\Delta t \delta_x[\tilde{\Phi} - \Phi(t)] = p_s u^{ex}(t+\Delta t), \tag{2.5.2.11}$$

$$p_s v(t+\Delta t) + 2\Delta t \delta_x[\tilde{\Phi} - \Phi(t)] = p_s v^{ex}(t+\Delta t), \tag{2.5.2.12}$$

$$p_s T(t+\Delta t) + 2\Delta t M_2[\tilde{D} - D(t)] = p_s T^{ex}(t+\Delta t). \tag{2.5.2.13}$$

Here $\Phi(t)$ and $D(t)$ have been computed using the explicit time integration over $2\Delta t$. Similar, for the pressure tendency we can write

$$P_s(t+\Delta t) + 2\Delta t N_1 \cdot [\tilde{D} - D(t)] = P^{ex}(t+\Delta t). \tag{2.5.2.14}$$

To find equations for the correction terms on the left hand side of equations (2.5.2.11-2.5.2.13), the divergence and geopotential equations (2.5.2.6-2.5.2.7) are then solved over the the small time-steps using

$$\begin{aligned} &[D(t+(n+1)\Delta\tau) - D(t)] - [D(t+(n-1)\Delta\tau) - D(t)] \\ &+ 2\Delta\tau(\delta_x^2 + \delta_y^2)[\Phi(t+n\Delta\tau) - \Phi(t)] \\ &= \frac{1}{m^i}[D_{ex}(t+\Delta t) - D(t-\Delta t)] \end{aligned} \tag{2.5.2.15}$$

and

$$
\begin{aligned}
&[\Phi(t+(n+1)\Delta\tau)-\Phi(t)]-[\Phi(t+(n-1)\Delta\tau)-\Phi(t)]\\
&+2\Delta\tau M_3[D(t+n\Delta\tau)-D(t)] \qquad .\\
&=\frac{1}{m^i}[\Phi_{ex}(t+\Delta t)-\Phi(t-\Delta t)]
\end{aligned}
\tag{2.5.2.16}
$$

The correction terms themselves are integrated in equations (2.5.2.15)and (2.5.2.16), and then added to equations (2.5.2.11-2.5.2.14).

$\Delta\tau$, the timestep of the fast modes, of course varies with the mode. For a clean separation of the modes, a vertical normal mode initialization developed and applied to the MM4/MM5 system by Errico (1986) is used at the beginning of the model run to calculate the vertical modes. In MM5, only the external and the fastest internal mode are being considered with different time steps. This allows the time-steps of the slow tendencies to be twice as large as they were with the previously used Brown-Campana (1978) algorithm, and they are comparable to the ones used in the nonhydrostatic numerics.

2.6 Lateral Boundary conditions for the coarsest mesh domain

2.6.1 Sponge Boundary Conditions

The sponge boundary condition is given by

$$\left(\frac{\partial \alpha}{\partial t}\right)_n = w(n)\left(\frac{\partial \alpha}{\partial t}\right)_{MC} + (1 - w(n))\left(\frac{\partial \alpha}{\partial t}\right)_{LS}, \tag{2.6.1}$$

where $n = 1, 2, 3, 4$ for cross-point variables, $n = 1, 2, 3, 4, 5$ for dot-point variables, α represents any variable, MC denotes the model calculated tendency, LS the large-scale tendency which is obtained either from observations or large-scale model simulations (one-way nesting), and n is the displacement in grid-points from the nearest boundary ($n = 1$ on the boundary). The weighting coefficients $w(n)$ for cross point variables (counting from the boundary points inward) are 0.0, 0.4, 0.7, and 0.9, while for dot-point variables they are equal to 0.0, 0.2, 0.55, 0.8, and 0.95. All other points in the coarse domain have $w(n) = 1$.

The above method cannot be used for the nonhydrostatic part of the model.

2.6.2 Nudging Boundary Conditions

The relaxation boundary condition involves "relaxing" or "nudging" the model-predicted variables toward a large-scale analysis. The method includes Newtonian and diffusion terms

$$\left(\frac{\partial \alpha}{\partial t}\right)_n = F(n)F_1(\alpha_{LS} - \alpha_{MC}) - F(n)F_2\Delta_2(\alpha_{LS} - \alpha_{MC}). \qquad n = 2, 3, 4 \tag{2.6.2}$$

F decreases linearly from the lateral boundary, such that

$$F(n) = \left(\frac{5 - n}{3}\right) \qquad n = 2, 3, 4, \tag{2.6.3}$$

$$F(n) = 0 \qquad n > 4, \qquad (2.6.4),$$

where F_1 and F_2 are given by

$$F_1 = \frac{1}{10\Delta t} \tag{2.6.5}$$

and

$$F_2 = \frac{\Delta s^2}{50\Delta t}. \tag{2.6.6}$$

This method is also used for the nonhydrostatic part of the model to nudge the pressure perturbation to the observations or larger-scale model simulations. However, for the nonhydrostatic solver the vertical velocity is not nudged. It can vary freely, except for the outermost rows and columns, where zero gradient conditions are specified. For the velocity components, the values at the inflow points are specified in a manner similar to the specification of temperature and pressure. The values at the outflow boundaries are obtained by extrapolation from the interior points. These boundary values are required only in the computation of the nonlinear horizontal momentum flux divergence terms; They are not required in the computation of the horizontal divergence.

2.6.3 Moisture variables

Cloud water, rain water, snow, and ice are considered zero on inflow and zero gradient on outflow. There is an option to specify the boundary values in the same way as for the other variables (e.g., these variables may be known in a one-way nesting application).

2.7 Upper radiative boundary condition

An option in the nonhydrostatic model is the upper radiative boundary condition. Klemp and Durran (1983) and Bougeault (1983) have developed an upper boundary condition that allows wave energy to pass through unreflected. It can be expressed for hydrostatic waves as

$$\hat{p} = \frac{\rho N}{K}\hat{w}, \tag{2.7.1}$$

where $\hat{p}$ and $\hat{w}$ are horizontal Fourier components of pressure and vertical velocity respectively, ρ and N are the density and buoyancy frequency near the model top, and K is the total horizontal wavenumber of the Fourier component. This expression should be enforced for all components if the energy transport is to be purely upward with no reflection.

The upper boundary condition is combined with the implicit pressure/vertical momentum calculation. Before either value at time $n+1$ is known, the values at the top model level (w_1 is staggered half a grid length above p_1) can be expressed as

$$p_1^{n+1} \; = \; b \; + \; a w_1^{n+1}, \tag{2.7.2}$$

where the coefficient $a(x, y, t)$ is dependent upon the thermodynamic structure and the bottom boundary condition on w in the model column. It varies within only 5 per cent of a constant value even with high terrain, and is also not strongly time-dependent. The value of $b(x,y,t)$ depends on pressure and most of the pressure tendency terms, and both a and b are known at this stage. So transforming, assuming a varies little about a non-zero constant and taking a mean value $\overline{a}$

$$\hat{p} \;=\; \hat{b} \;+\; \overline{a}\hat{w}. \tag{2.7.3}$$

Combining (2.7.3) with the radiative condition (2.7.1) for wavenumber $K = 2\pi/\lambda$, taking $\overline{\rho N}$ at the top of the model, and eliminating $\hat{p}$, gives

$$\hat{w} \;=\; \frac{K\hat{b}}{\overline{\rho N} \;-\; \overline{a}K}. \tag{2.7.4}$$

Using a limited-area 2D cosine transform, the forward transform, multiplication and backward transform can be combined into a single operator on the b field to give w_1^{n+1}. Hence

$$w_{IJ} \;=\; \sum_{i=I-6}^{I+6}\sum_{j=J-6}^{J+6} \alpha_{ij} b_{ij}, \tag{2.7.5}$$

where we have localized the transform to 13×13 points, and array α can be precalculated and kept constant for the time integration. The elements of α are found from

$$\alpha_{ij} \;=\; \sum_{k=0}^{6}\sum_{l=0}^{6} \frac{\delta_i \delta_j \delta_k \delta_l}{36} \cos\frac{2\pi ki}{12} \cos\frac{2\pi lj}{12} f(K), \tag{2.7.6}$$

with $f(K) = \frac{K}{\overline{\rho N} - \overline{a}K}$ and $K = (\hat{k}^2 + \hat{l}^2)^{\frac{1}{2}}$. $\delta = 1$ except for limits of summations where $\delta = \frac{1}{2}$.

Following the suggestion of Klemp and Durran, the finite differencing of pressure gradients and divergences should be taken into account in defining the effective wavenumbers. For a B-grid staggering, the effective wavenumbers can be expressed in terms of the dimensionless wavenumbers, k and l, where

$$\hat{k} \;=\; \frac{2}{\Delta x} \sin\frac{k\pi}{12} \cos\frac{l\pi}{12}, \tag{2.7.7a}$$

$$\hat{l} \;=\; \frac{2}{\Delta x} \sin\frac{l\pi}{12} \cos\frac{k\pi}{12}, \tag{2.7.7b}$$

and Δx is the grid length.

The scheme is summarized as follows; by the precalculation of parameters $\overline{a}$ and $\overline{\rho N}$ for the model domain, use of (2.7.6) to precalculate coefficients α, then implementation of (2.7.5) during the simulation.