Appendix A: Derivation of Basic MM5 Equations

A.1 Derivation of Thermodynamic Equation

First Law of Thermodynamics

$$dQ = c_v dT + p d\alpha = c_p dT - \alpha dp \qquad \text{(A.1)}$$

since from the gas law $R dT = p d\alpha + \alpha dp$ and $c_p - c_v = R$. Temperature tendency therefore is given by

$$c_p \frac{DT}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt} + \dot{Q} \qquad \text{(A.2)}$$

A.2 Derivation of Pressure Tendency Equation

From Gas Law

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{T}\frac{DT}{Dt} \qquad \text{(A.3)}$$

Continuity and Thermodynamics lead to

$$\frac{1}{p}\frac{Dp}{Dt} = -\nabla \cdot \mathbf{v} + \frac{\dot{Q}}{c_p T} + \frac{1}{c_p \rho T}\frac{Dp}{Dt} \qquad \text{(A.4)}$$

However, $c_p \rho T = \left(\frac{c_p}{R}\right) p$, so

$$\frac{1}{p}\frac{Dp}{Dt}\left(1 - \frac{R}{c_p}\right) = -\nabla \cdot \mathbf{v} + \frac{\dot{Q}}{c_p T} \qquad \text{(A.5)}$$

But $1 - \frac{R}{c_p} = \frac{c_v}{c_p} = \frac{1}{\gamma}$, therefore

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{v} + \frac{\gamma p \dot{Q}}{c_p T} \qquad \text{(A.6)}$$

A.3 Forms of the Vertical Momentum Equation

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = D_w \qquad \text{(A.7)}$$

Defining $\alpha = \frac{1}{\rho}$,

$$\frac{Dw}{Dt} + \alpha\frac{\partial p}{\partial z} + g = D_w \qquad \text{(A.8)}$$

Defining hydrostatic reference and perturbation, $\alpha = \alpha_0 + \alpha'$, $p = p_0 + p'$,

$$\frac{Dw}{Dt} + (\alpha_0 + \alpha')\left(\frac{\partial p_0}{\partial z} + \frac{\partial p'}{\partial z}\right) + g = D_w \qquad \text{(A.9)}$$

By definition, $\alpha_0\frac{\partial p_0}{\partial z} = -g$, so

$$\frac{Dw}{Dt} + \alpha'\frac{\partial p'}{\partial z} + \alpha_0\frac{\partial p'}{\partial z} + \alpha'\frac{\partial p_0}{\partial z} = D_w \qquad \text{(A.10)}$$

which can be written as

$$\frac{Dw}{Dt} + \alpha\frac{\partial p'}{\partial z} - g\frac{\alpha'}{\alpha_0} = D_w \qquad \text{(A.11)}$$

This can be expanded as

$$\frac{Dw}{Dt} + \alpha\frac{\partial p'}{\partial z} - g\frac{\alpha - \alpha_0}{\alpha_0} = D_w \qquad \text{(A.12)}$$

In terms of ρ, this is

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p'}{\partial z} - g\frac{\frac{1}{\rho} - \frac{1}{\rho_0}}{\frac{1}{\rho_0}} = D_w \qquad \text{(A.13)}$$

which is

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p'}{\partial z} + g\frac{\rho'}{\rho} = D_w \tag{A.14}$$

This can be expressed in terms of temperature and pressure perturbations for the buoyancy term because

$$-\frac{\rho'}{\rho} = \frac{\rho_0}{\rho} - 1 = \frac{p_0 T}{p T_0} - 1 = \frac{p_0}{p}\left(\frac{T}{T_0} - \frac{p}{p_0}\right) = \frac{p_0}{p}\left(\frac{T'}{T_0} - \frac{p'}{p_0}\right) \tag{A.15}$$

So

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p'}{\partial z} - g\frac{p_0}{p}\left(\frac{T'}{T_0} - \frac{p'}{p_0}\right) = D_w \tag{A.16}$$

A.4 Coordinate Transformation

General coordinate transformation $(x, y, z) \rightarrow (x, y, \sigma)$

$$\left(\frac{\partial}{\partial x}\right)_z \rightarrow \left(\frac{\partial}{\partial x}\right)_\sigma - \left(\frac{\partial z}{\partial x}\right)_\sigma \frac{\partial}{\partial z} \tag{A.17}$$

but $\delta z = \dfrac{-\delta p_0}{\rho_0 g} = -\dfrac{(p^* \delta\sigma + \sigma \delta p^*)}{\rho_0 g}$, so

$$\left(\frac{\partial}{\partial x}\right)_z \rightarrow \left(\frac{\partial}{\partial x}\right)_\sigma - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial x}\frac{\partial}{\partial \sigma} \tag{A.18}$$

A.5 Derivation of $\dot{\sigma}$ Relation

$$\sigma = \frac{p_0 - p_{top}}{p_{surf} - p_{top}} = \frac{p_0 - p_{top}}{p^*} \tag{A.19}$$

where p_{top} and p_{surf} are the values of p_0 at the top and surface and $p^* = p_{surf} - p_{top}$.

$$\dot{\sigma} = \frac{D\sigma}{Dt} \tag{A.20}$$

Therefore

$$\dot{\sigma} = \frac{1}{p^*}\frac{Dp_0}{Dt} - \frac{(p_0 - p_{top})}{(p^*)^2}\frac{Dp^*}{Dt} \tag{A.21}$$

Expanding total derivatives noting that $p_0 = p_0(z)$ and $p^* = p^*(x,y)$ and also that p_0 is hydrostatic

$$\dot{\sigma} = -\frac{\rho_0 g}{p^*}w - \frac{\sigma}{p^*}\left(u\frac{\partial p^*}{\partial x} + v\frac{\partial p^*}{\partial y}\right) \tag{A.22}$$