Energy Conservation in Compressible Nonhydrostatic Solvers

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Consensus from meetings about weather/climate model:

Required features:

- "The solver should integrate the fully compressible nonhydrostatic equations of motion"
  - Nonhydrostatic: for simulating deep convection and other weather systems
  - Compressible: meaning un-approximated equations (valid at all scales, all conditions)
- "The solver should conserve mass"
  - Required for long-term simulations (months-to-years), and for modeling of chemical transport/dispersion

Highly desired features:

- Energy conservation

Compressible Nonhydrostatic Modeling: Historical Development

- Historically (pre-2000), compressible nonhydrostatic models did not enforce conservation of any properties (e.g., total mass, energy, momentum)
- Reasons include:
  - simplicity (less memory overhead)
  - efficiency (faster)
  - short integration times (hours or days)
  - importance? (no clear impact on solutions of interest)

Compressible Nonhydrostatic Modeling: Recent Developments

- Mass conservation has recently become a primary design feature of some compressible nonhydrostatic modeling systems (e.g., WRF Model developed at NCAR)
- Reasons:
  - transport and dispersion applications
  - longer integration times (e.g., regional climate modeling)
  - importance in certain weather systems has become clear (e.g., hurricane intensity)

Compressible Nonhydrostatic Modeling: Unresolved Issues

- Energy conservation has rarely been enforced
- Primary reason: complexity!
  - Dissipative heating:
    - from subgrid turbulence
    - from high-order diffusion
    - from PBL parameterization
    - difficult to maintain in constantly evolving community models
  - Moist processes:
    - sedimentation of hydrometeor energy
    - dissipative heating around falling hydrometeors
    - debate about exact form of moist equations (multiphase flow interactions)

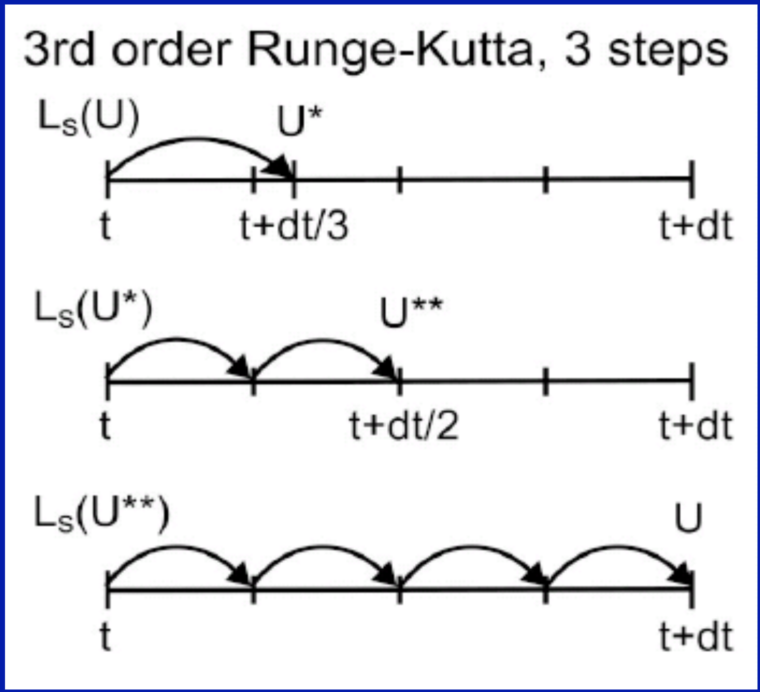
Goals of this study

- Determine how conservation of total energy can be incorporated into the WRF Model's solver
  - at what cost? (e.g., execution time, accuracy, complexity)
  - for what benefit?
- Split-explicit time integration:
  - Integrate terms responsible for propagation of acoustic modes (and gravity waves) on a "small" timestep
  - Integrate all other terms (e.g., advection, diffusion, moist processes, radiation) on a "large" timestep

From: WRF Model Tutorial notes

(Skamarock and Dudhia, 2008)

Runge-Kutta (e.g., WRF):



A traditional approach:

Integrate equations for pressure (π), velocity (u,w), and potential temperature (θ):

$$\pi = \left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}, \qquad T = \theta\pi$$

$$\alpha(x,y,z,t) = \overline{\alpha}(z) + \alpha'(x,y,z,t)$$

$$\frac{\partial u}{\partial t} + c_p\theta\frac{\partial \pi'}{\partial x} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} + F_u$$

$$\frac{\partial w}{\partial t} + c_p\theta\frac{\partial \pi'}{\partial z} - g\frac{\theta'}{\overline{\theta}} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} + F_w$$

$$\frac{\partial \pi'}{\partial t} + \pi\frac{R}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = -u\frac{\partial \pi}{\partial x} - w\frac{\partial \pi}{\partial z} + F_\pi$$

$$\frac{\partial \theta'}{\partial t} + w\frac{\partial \overline{\theta}}{\partial z} = -u\frac{\partial \theta'}{\partial x} - w\frac{\partial \theta'}{\partial z} + F_\theta$$

small time steps ⟵ ⟶ large time steps

- Primary problem: conservation is not guaranteed
- "Conservation" is defined as both:
  - global conservation of a fundamental variable (such as total mass, total momentum, and total energy)
  - local conservation during application of a numerical algorithm … e.g., finite volume method (flux of mass out of a control volume = flux of mass into a neighboring control volume)
- Solution: integrate conservation equations (e.g., for mass, momentum, etc), and use finite volume methods

Conservative equations:

Integrate equations for density (ρ), momentum ($U = \rho u$, $W = \rho w$), and entropy ($\Theta = \rho\theta$):

$$p = p_0 \left(\frac{R\Theta}{p_0} \right)^{\frac{c_p}{c_v}}$$

$$\frac{\partial U}{\partial t} + \frac{c_p}{c_v} R\pi \frac{\partial \Theta'}{\partial x} = -\frac{\partial U u}{\partial x} - \frac{\partial W u}{\partial z} + F_U$$

$$\frac{\partial W}{\partial t} + \frac{c_p}{c_v} R\pi \frac{\partial \Theta'}{\partial z} + g \left(\rho' - \overline{\rho} \frac{\pi'}{\overline{\pi}} \right) = -\frac{\partial U w}{\partial x} - \frac{\partial W w}{\partial z} + F_W$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0$$

$$\frac{\partial \Theta'}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial W\theta}{\partial z} = F_\Theta$$

Solution: recast momentum variables in terms of perturbations from a recent time t:

$$U = U^t + U''$$
$$W = W^t + W''$$

$$\frac{\partial U''}{\partial t} + \frac{c_p}{c_v} R\pi \frac{\partial \Theta'}{\partial x} = -\frac{\partial U^t u}{\partial x} - \frac{\partial W^t u}{\partial z} + F_U$$

$$\frac{\partial W''}{\partial t} + \frac{c_p}{c_v} R\pi \frac{\partial \Theta'}{\partial z} + g\left(\rho' - \overline{\rho}\frac{\pi'}{\overline{\pi}}\right) = -\frac{\partial U^t w}{\partial x} - \frac{\partial W^t w}{\partial z} + F_W$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U''}{\partial x} + \frac{\partial W''}{\partial z} = -\frac{\partial U^t}{\partial x} - \frac{\partial W^t}{\partial z}$$

$$\frac{\partial \Theta'}{\partial t} + \frac{\partial U''\theta}{\partial x} + \frac{\partial W''\theta}{\partial z} = -\frac{\partial U^t\theta}{\partial x} - \frac{\partial W^t\theta}{\partial z} + F_\Theta$$

→ In this system, total mass is conserved (locally and globally)

Klemp et al. (2007, MWR)

Unresolved issues:

- No guarantee of momentum conservation, owing to form of pressure-gradient terms
- No guarantee of total energy conservation
  - In fact, dissipative heating is typically neglected
  - Dissipative heating is known to be important at high wind speeds (e.g., hurricanes) and long-term integrations (seasonal time scales)

Use total energy, E_t, as a predicted variable:

$$\underbrace{E_t}_{\text{total}} = \underbrace{\rho c_v T}_{\text{internal }(e)} + \underbrace{\rho g z}_{\text{potential }(\Phi)} + \underbrace{\rho \frac{1}{2}\left(u^2 + w^2\right)}_{\text{kinetic }(K)}$$

Pressure gradient in terms of E_t:

$$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{R}{c_v}\left(E_t - \Phi - K\right)\right)$$

Integrate a governing equation for E_t:

$$\frac{\partial E_t}{\partial t} = -\frac{\partial U e^*}{\partial x} - \frac{\partial W e^*}{\partial z}$$

where

$$e^* \equiv \frac{E_t + p}{\rho}$$

New solution procedure:
use same techniques as before

$$\frac{\partial U''}{\partial t} + \frac{\partial}{\partial x}\left(\frac{R}{c_v}\left(E_t' - \rho' g z\right)\right) = -\frac{\partial U^t u}{\partial x} - \frac{\partial W^t u}{\partial z} + \frac{\partial}{\partial x}\left(\frac{R}{c_v}K\right)$$

$$\frac{\partial W''}{\partial t} + \frac{\partial}{\partial z}\left(\frac{R}{c_v}\left(E_t' - \rho' g z\right)\right) + g\rho' = -\frac{\partial U^t w}{\partial x} - \frac{\partial W^t w}{\partial z} + \frac{\partial}{\partial z}\left(\frac{R}{c_v}K\right)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U''}{\partial x} + \frac{\partial W''}{\partial z} = -\frac{\partial U^t}{\partial x} - \frac{\partial W^t}{\partial z}$$

$$\frac{\partial E_t'}{\partial t} + \frac{\partial U'' e^*}{\partial x} + \frac{\partial W'' e^*}{\partial z} = -\frac{\partial U^t e^*}{\partial x} - \frac{\partial W^t e^*}{\partial z}$$

Note:

- All terms are in flux form (except buoyancy) $\rightarrow$ conservation
- All variables on left side are either held fixed (e^*) or are integrated on the small steps $\rightarrow$ efficiency and simplicity

Tests

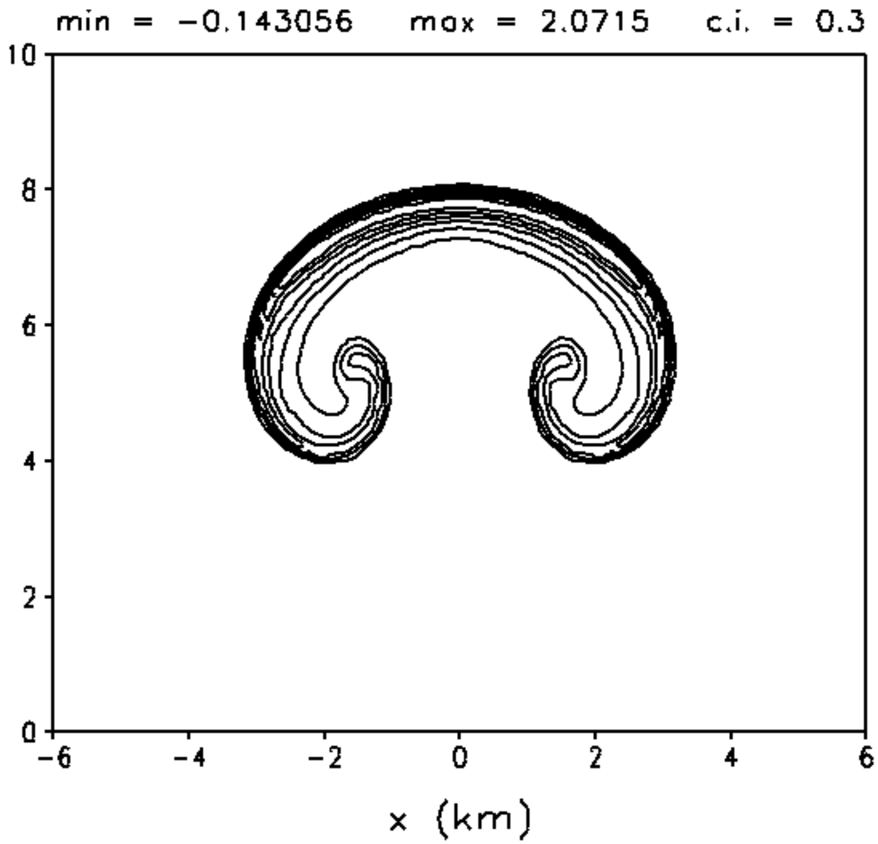
- Developed a prototype code that can integrate all three equation sets:
  - Non-conserving (u, w, π, θ)
  - Mass-conserving (U, W, ρ, Θ)
  - Mass,Momentum,Energy-conserving (U, W, ρ, E_t)
- Same techniques as WRF Model (ARW):
  - 3rd-order Runge-Kutta w/ time-splitting
  - 5th-order advection operators (finite-volume-form)
  - Cartesian height coordinate

A simple test:

- Warm bubble ("moist benchmark") case used by Bryan and Fritsch (2002, MWR)
- No analytic solution, but:
  - well resolved (does not collapse to grid-scale)
  - well-known solution (produced by many models)
  - useful for testing equation sets
- Details:
  - 2D, $\Delta x = \Delta z = 100$ m
  - Statically neutral initial state with warm bubble
  - Integrate for 1000 s

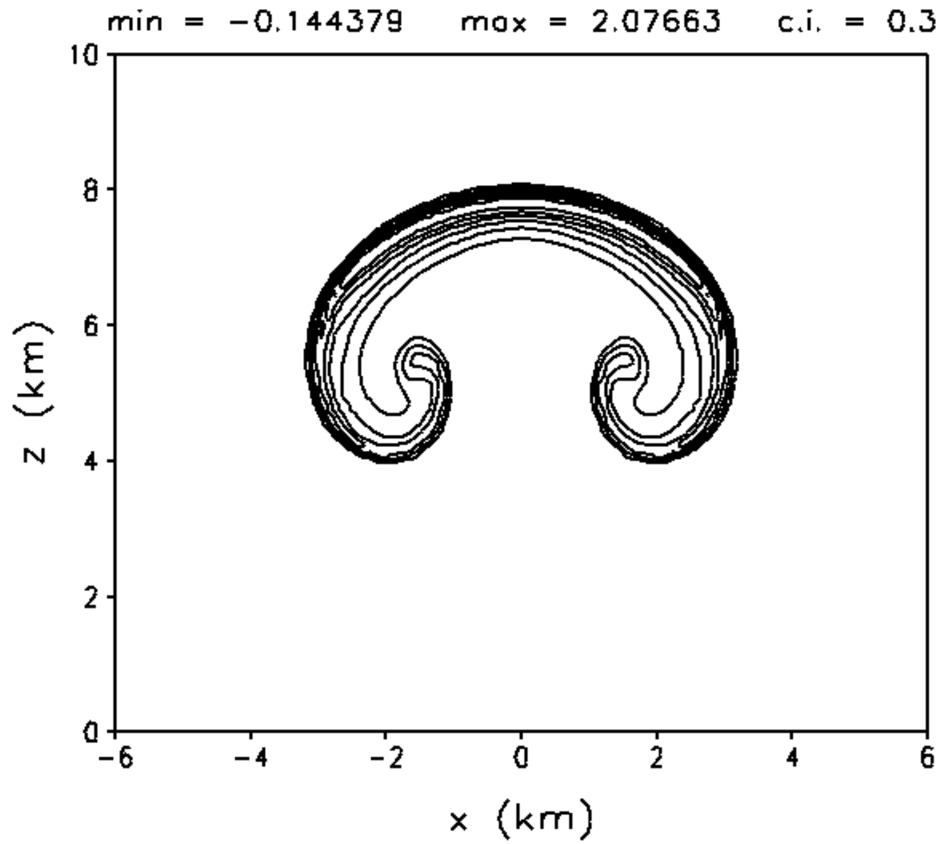
θ′ (K) at t = 1000 s

Non-conserving



run time: 79 s

Mass-conserving

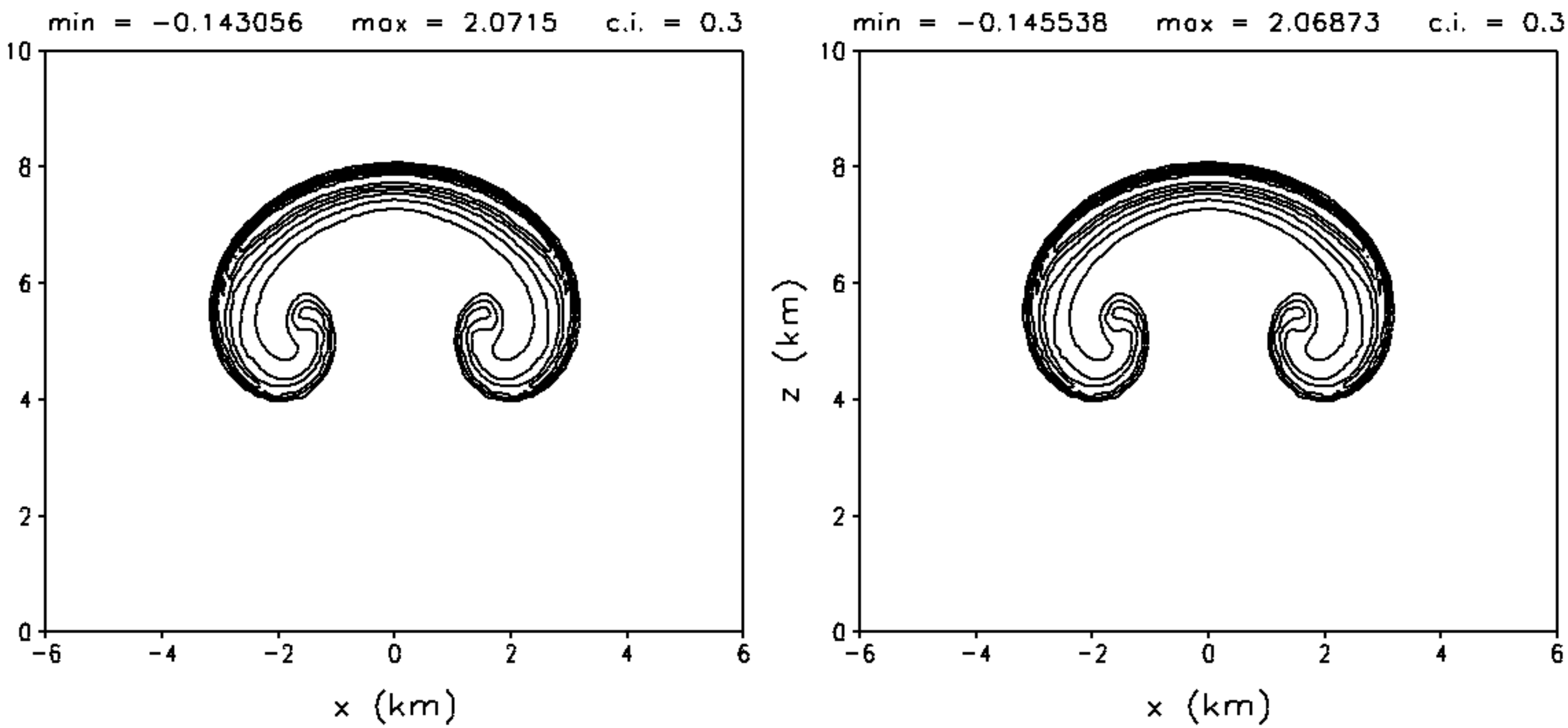


run time: 89 s

θ′ (K) at t = 1000 s

Non-conserving

Mass,Mo,Ene-conserving

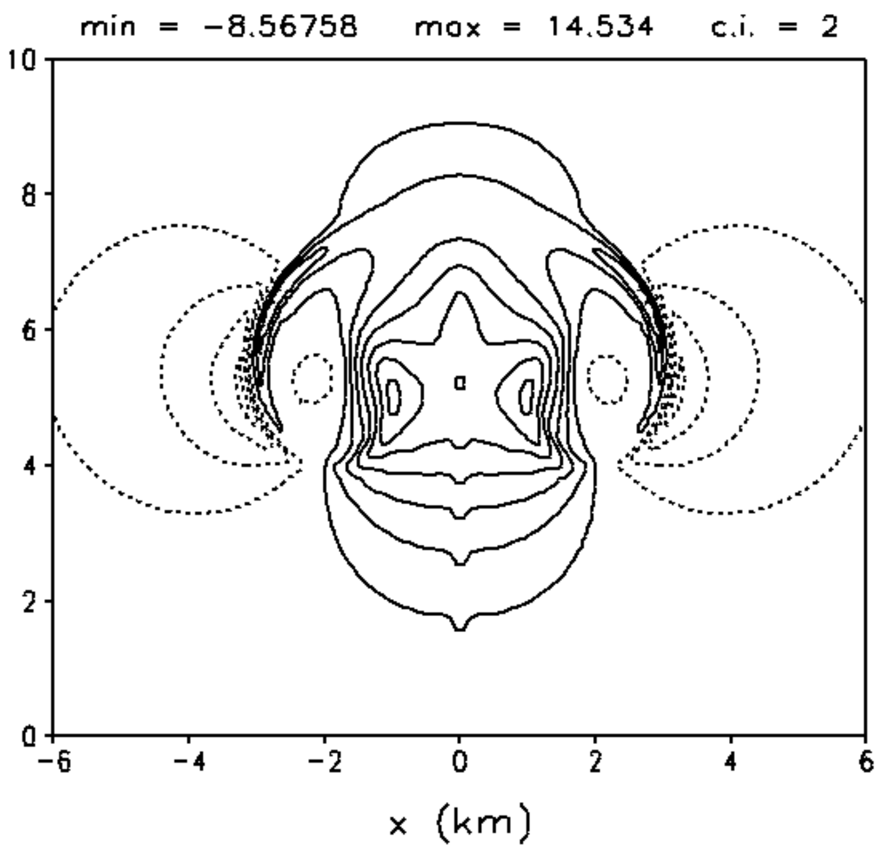


run time: 79 s

run time: 82 s

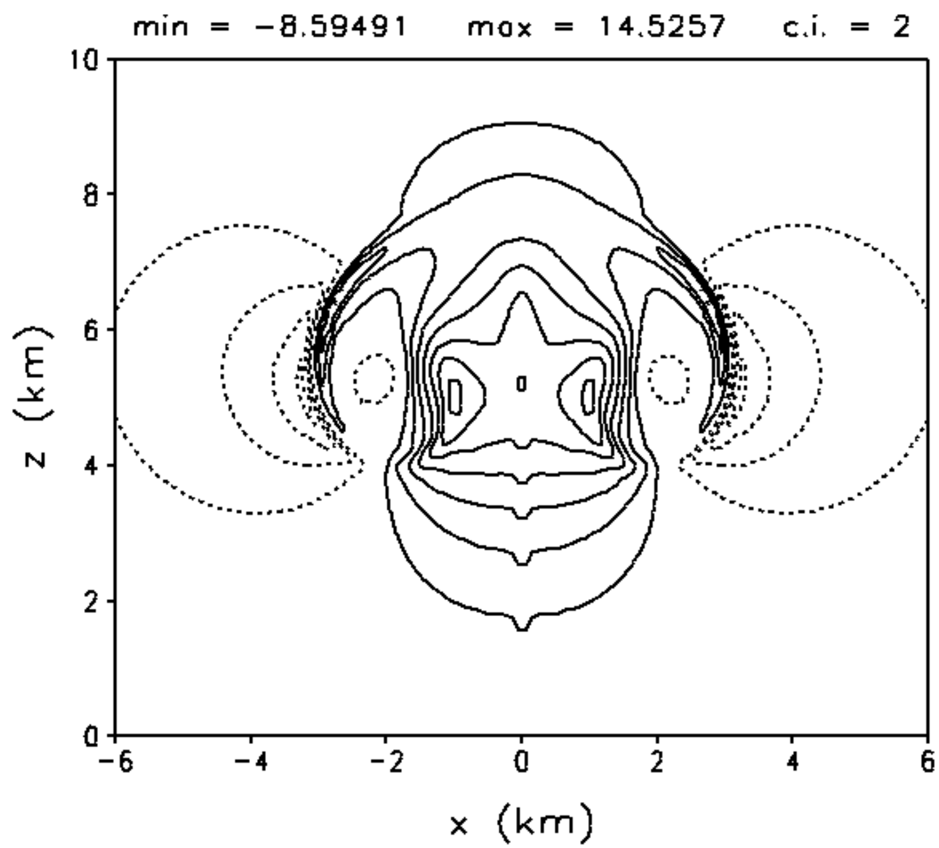
w (m/s) at t = 1000 s

Non-conserving



run time: 79 s

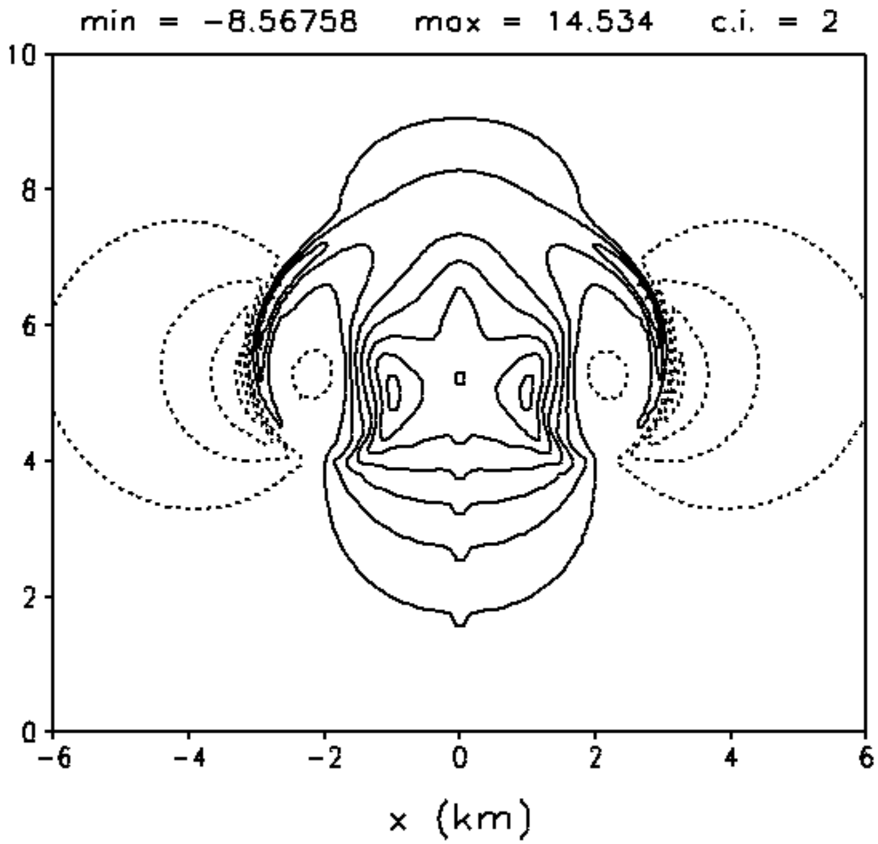
Mass-conserving



run time: 89 s

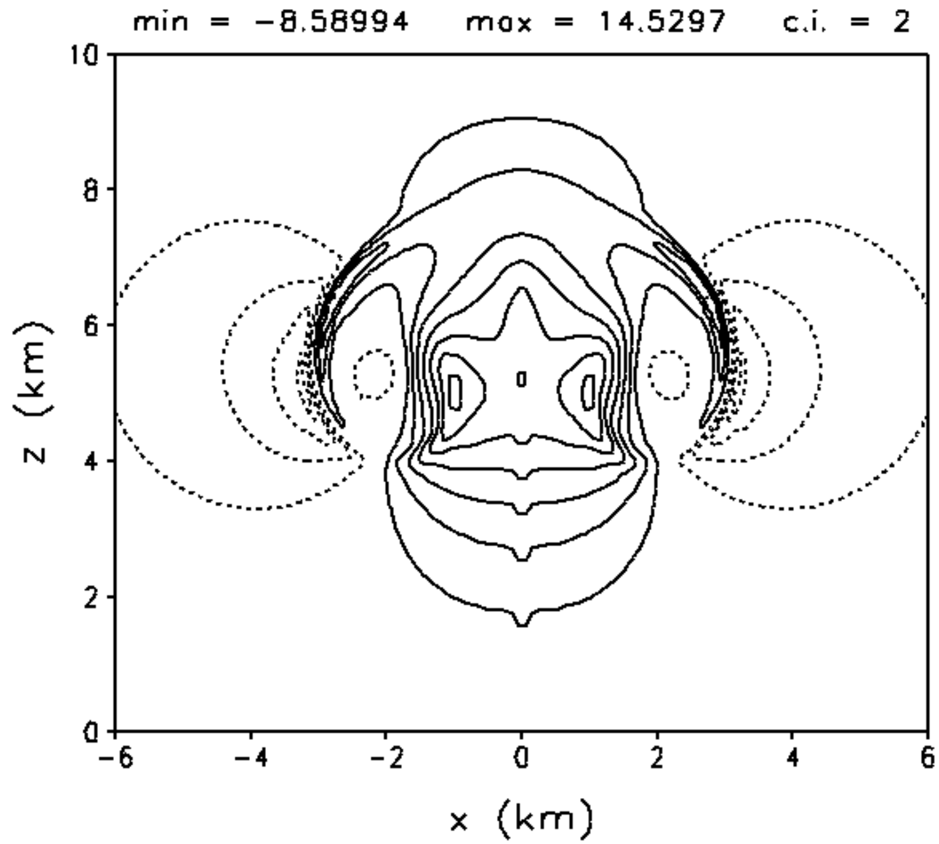
w (m/s) at t = 1000 s

Non-conserving



run time: 79 s

Mass,Mo,Ene-conserving



run time: 82 s

Efficiency of dry bubble tests

- Run times:
  - Non-conserving: 79 s
    (fewer terms on small steps)
  - Mass-conserving: 89 s
    (more terms on small steps)
    (calculation of π is expensive)
  - Mass,Mo,Ene-conserving: 82 s
    (more terms on small steps)

Diabatic heating terms in entropy equation:

$$\frac{\partial \Theta}{\partial t} = \cdots + \dot{\Theta}$$

Total energy equation:

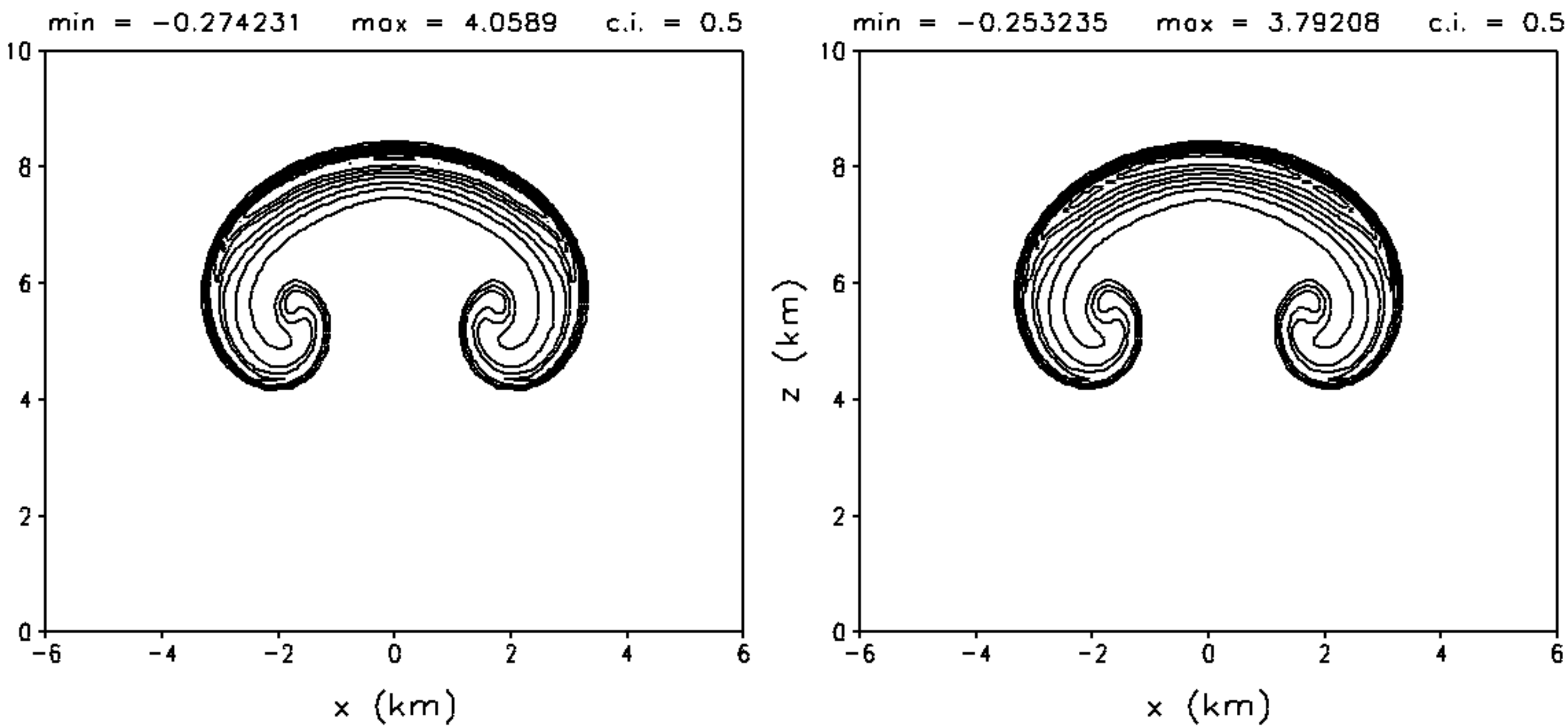
$$\frac{\partial \left(e + \Phi + K\right)}{\partial t} = \cdots + 0$$

$$e \equiv \rho c_v T + \rho_v c_{vv} T + \rho_l (c_l T - L^*)$$

→ No special treatment needed for diabatic heating on small timesteps

Saturated case: θ_e' (K) at t = 1000 s

Non-conserving | Mass,Mo,Ene-conserving



run time: 111 s | run time: 121 s

Examine viscous terms in momentum equations:

$$\frac{\partial U_i}{\partial t} = \cdots + \frac{\partial \tau_{ij}}{\partial x_j}$$

Kinetic energy equation:

$$\frac{\partial K}{\partial t} = \cdots + \frac{\partial \left(u_i \tau_{ij}\right)}{\partial x_j} - \tau_{ij}\frac{\partial u_i}{\partial u_j}$$

Internal energy equation:

$$\frac{\partial e}{\partial t} = \cdots + \tau_{ij}\frac{\partial u_i}{\partial u_j}$$

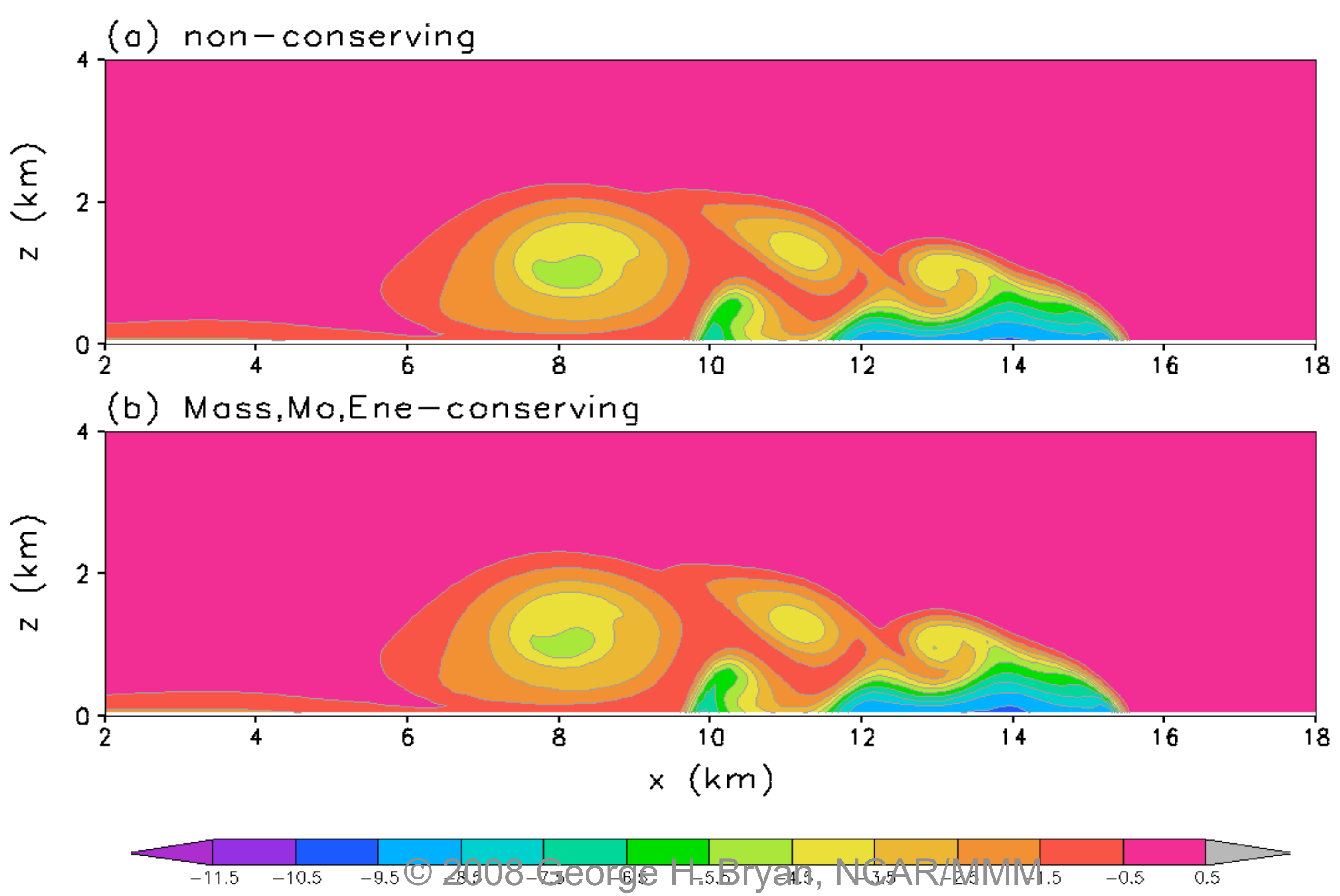
Total energy equation:

$$\frac{\partial E_t}{\partial t} \equiv \frac{\partial \left(e + \Phi + K\right)}{\partial t} = \cdots + 0$$

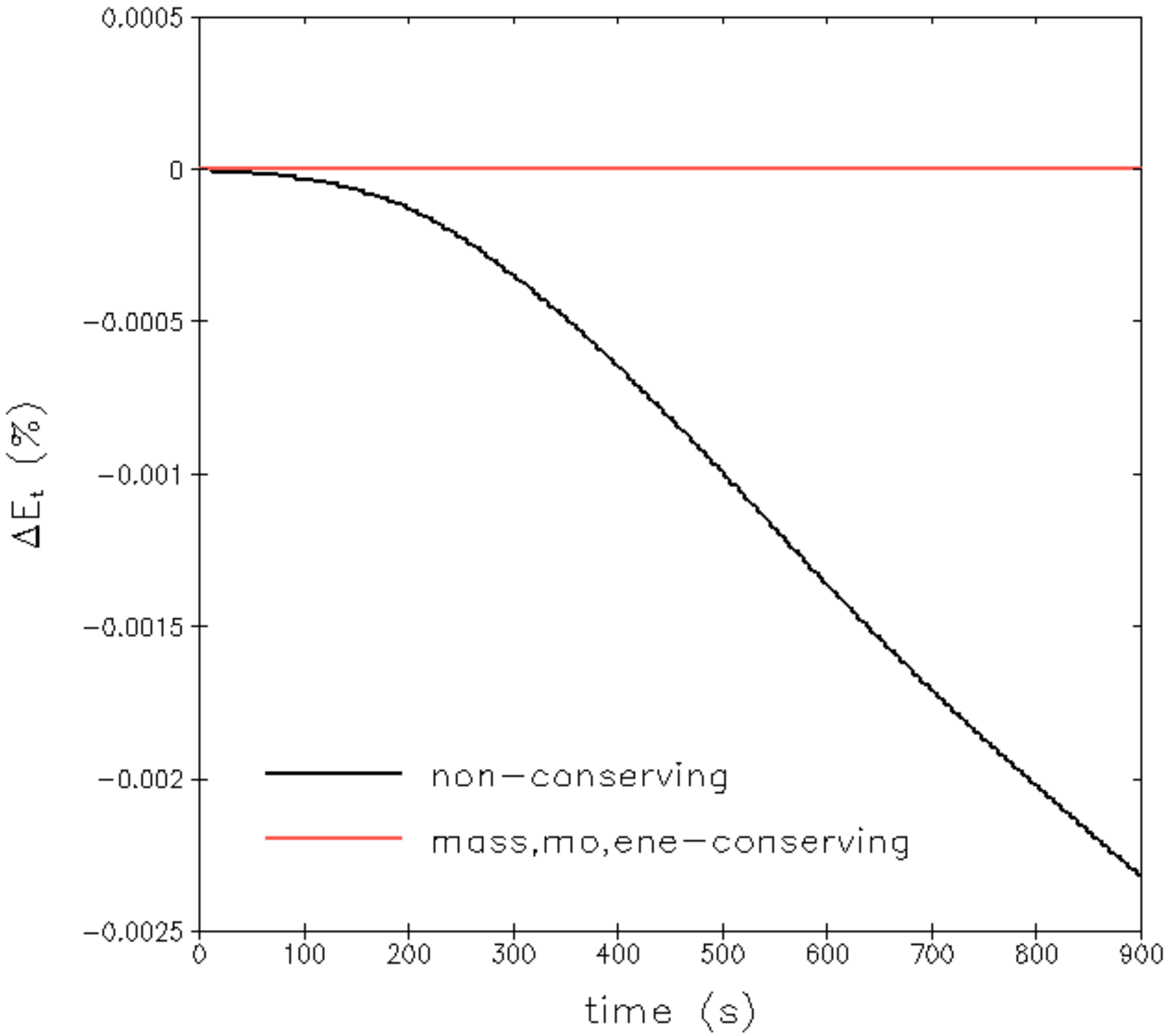
→ No extra work needed to account for dissipative heating

Gravity current test case

- 2d, isentropic environment
- constant diffusion coefficient (75 m^2 s^{-1})
- Δ = 100 m:

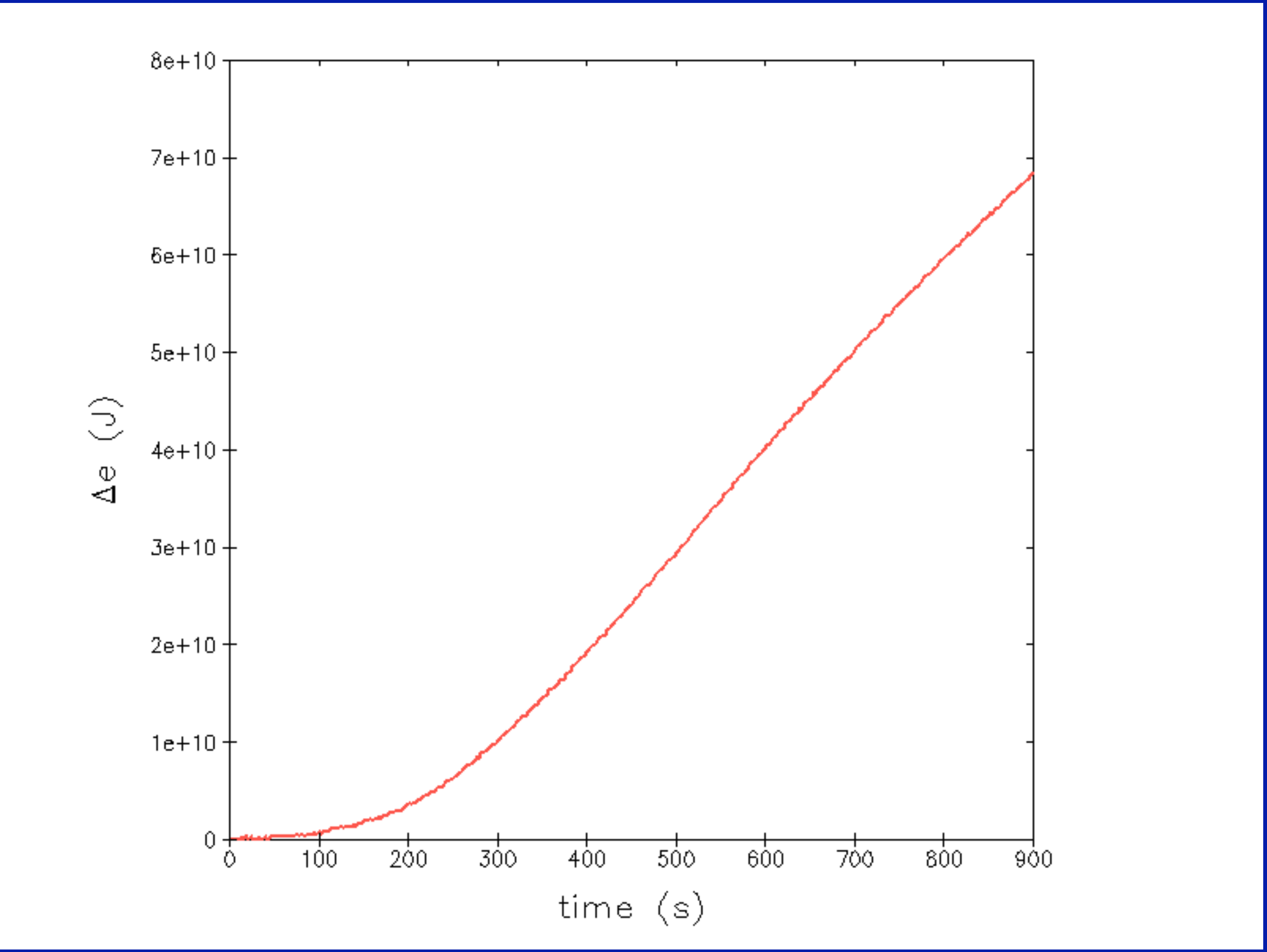


Change in total energy:



→ old solver loses energy over time
(owing to lack of dissipative heating)

Difference in internal energy:



Summary

- It is possible to formulate a compressible nonhydrostatic solver that conserves (locally and globally) total mass, momentum, and energy
  - only a small increase in cost
  - some clear benefits
- Unresolved topics / points for discussion:
  - Complicated moisture terms (mostly related to differential fall velocity in multiphase flows)
  - Integration on the sphere?
  - Relative merits of conserving other properties
    - Higher-order quantities (potential enstrophy, tracer variance), potential vorticity?
  - Relative merits of conservation vs numerical implementation
    - e.g., vector-invariant form of momentum equations