Evaluation of a Time-Dependent Model for the Intensification of Tropical Cyclones

KE PENG

Key Laboratory for Mesoscale Severe Weather, Ministry of Education, and School of Atmospheric Science, Nanjing University, Nanjing, China, and National Center for Atmospheric Research, Boulder, Colorado

RICHARD ROTUNNO AND GEORGE H. BRYAN

National Center for Atmospheric Research, Boulder, Colorado

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ABSTRACT

Axisymmetric and three-dimensional simulations are used to evaluate the theory of tropical cyclone (TC) intensification proposed by K. A. Emanuel, which is based on gradient wind balance and moist-neutral ascent along angular momentum (M) surfaces. According to the numerical model results, the intensification of the TC can be divided into two periods, phase I and phase II. During phase I, the TC intensifies while the M and saturation entropy (s^*) surfaces evolve from nearly orthogonal to almost congruent. During phase II, the M and s* surfaces in the eyewall and outflow are congruent as the TC intensifies, which is consistent with Emanuel's study. Therefore, the condition of moist slantwise neutrality in Emanuel's study is sufficiently satisfied throughout the intensification in phase II. It is also found that the sensitivity of the intensification rates to the surface exchange coefficient for entropy C_k matches Emanuel's theoretical result, which is that the intensification rate is proportional to C_k . However, the intensification rate varies in proportion to the surface exchange coefficient for momentum C_d , while the Emanuel model growth rate is insensitive to C_d . Furthermore, although the tendency diagnosed by Emanuel is qualitatively similar to the numerical model result during phase II, it is not quantitatively similar. The present analysis finds the inclusion of non-gradient wind effects in the theoretical framework of Emanuel's study produces an intensification rate that is quantitatively similar to the numerical model results. The neglect of non-gradient wind effects in Emanuel's study may be the reason for the different dependence of its intensification rate on C_d compared to that of the numerical model. Other aspects of Emanuel's study in the context of recent research on TC intensification are discussed.

1. Introduction

In two recent review papers, Montgomery and Smith (2014, 2017) describe the progression of ideas in roughly the past half century concerning the intensification of tropical cyclones (TCs). Restricting attention to the idealized axisymmetric TC, one can see that all theories 1) make a conceptual separation between an interior region and a boundary layer and 2) attempt to reconcile the disparate dynamics of smaller-scale cumulus convection with the larger-scale TC circulation. The focus of the present paper is the recent model for axisymmetric TC intensification developed in Emanuel (2012, hereafter E12), which has the advantage of possessing an approximate analytical solution. Specifically, we use a cloud-resolving axisymmetric nonhydrostatic model, set

up to follow closely the physical content of the E12 model, to assess the latter's predictions and to further clarify its dynamics. Of particular interest here are the effects of the surface exchange coefficients on TC intensification as predicted by the E12 model.

Emanuel (1986, hereafter E86) developed a steadystate axisymmetric TC model in which the interior is assumed to be in hydrostatic and gradient wind balance, and convective heat transport (saturated moist entropy transport s^*) occurs along angular momentum (*M*) surfaces; the boundary layer supplies a relation between s^* and *M*. A basic result of the E86 model is that of the maximum tangential wind, $V_{\text{max}} \sim (C_k/C_d)^{1/2}$ where C_k and C_d are the surface exchange coefficients for entropy and momentum, respectively. The E86 model forms the basis for the time-dependent theoretical model in Emanuel (1997, hereafter E97) and the improved version in E12. This time-dependent system consists of

Corresponding author: Dr. Richard Rotunno, rotunno@ucar.edu

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three main equations, formulated in *M* coordinates. As in E86, hydrostatic and gradient wind balance in the TC interior and moist neutrality along M surfaces are used to relate the tangential wind speed V at the top of boundary layer to the M gradient of s^* and the difference between the boundary layer and outflow temperatures. The second equation is a diagnostic equation for outflow temperature, which is based on the selfstratification theory¹ proposed by Emanuel and Rotunno (2011, hereafter ER11). The third equation of the E12 model is the boundary layer equation, which predicts the evolution of s^* in M coordinates. Thus, the time change of TC intensity in E12 results from an imbalance between the effects of sea surface entropy and momentum fluxes. According to the solution of the approximate system of E12 [his (19)], the intensification rate is proportional to C_k and inversely proportional to C_d .²

Craig and Gray (1996), using a nonhydrostatic axisymmetric cloud model, found that the intensification rate increases with increasing values of the exchange coefficients for heat and moisture, which is qualitatively consistent with the E12 model. However, they found that the intensification rate is relatively insensitive to changes in C_d .

Rosenthal (1971) used an axisymmetric, multilevel primitive equation model with a modified Kuo cumulus parameterization scheme to examine the dependence of the intensification rate on C_d . He found that increases in C_d lead to faster growth rates (but weaker peak intensities). In a recent study, Montgomery et al. (2010) conducted idealized three-dimensional numerical simulations to investigate the sensitivity of the TC intensification rate to changes in the surface drag coefficient. They also found the TC intensification rate increases with increasing C_d until a certain threshold value is reached, beyond which the TC intensification rate is relatively insensitive to further increases in C_d . This latter finding may explain the insensitivity of intensification rate to increases in C_d found in Craig and Gray (1996), as their values of C_d (which included the full increase with wind speed) were relatively large. Given the somewhat contrasting results to be found among numerical studies and the theoretical model of E12, this study uses an axisymmetric, nonhydrostatic cloud model to study the sensitivity of TC behavior to changes in C_k and C_d varied independently.

The paper is organized as follows. In section 2 we describe the numerical simulations used here to evaluate the E12 theoretical model. In section 3 the numerical simulations are compared to the numerical integrations of the E12 model and certain theoretical predictions are tested, especially with reference to the effects of C_k and C_d on TC intensification. In section 4 we seek to clarify aspects of the E12 model by transforming the numerical solutions of E12 from angular momentum to cylindrical coordinates. In section 5 a series of 3D simulations with more-standard physical parameterizations were conducted to check whether the primary results of the idealized axisymmetric simulations remain applicable. Section 6 contains a summary and conclusions.

2. Numerical simulations

Bryan and Rotunno (2009a, hereafter BR09a) compared the maximum intensity of numerically simulated TCs with the E86 theoretical estimate for maximum potential intensity at steady state. Following BR09a, we use here the axisymmetric, nonhydrostatic Cloud Model, version 1 (CM1), as described in Bryan and Rotunno (2009b, hereafter BR09b), to perform a series of numerical experiments examining the intensification rate of the simulated TCs. Most of the model settings used in the present simulations are identical to those used in BR09a. The initial environmental sounding is saturated and was constructed assuming constant pseudoadiabatic equivalent potential temperature as in BR09a (see their Fig. 1). As in BR09a, we employ a small horizontal turbulence length scale $(l_h = 94 \text{ m})$ and immediately remove condensate in excess of $0.1 \,\mathrm{g \, kg^{-1}}$. These settings are chosen to provide a direct comparison to the idealized framework of E12. The initial vortex is identical to that in Rotunno and Emanuel (1987, hereafter RE87) except here the initial maximum tangential wind speed is $22.5 \,\mathrm{m \, s^{-1}}$. For all the experiments, dissipative heating is not included and the surface exchange coefficients are set to be constant. The model domain is 1500 km in radial direction with a radial grid spacing of 1 km for r < 64 km and a stretched radial resolution for $r \ge 64$ km. The vertical grid spacing varies from 50 to 500 m for z < 5 km and becomes 500 m from 5 to 25 km. The stretched grid resolution is believed to be adequate for resolving the boundary layer and inner-core dynamics, which are very important in E12's theory.

¹ The theory establishes the functional dependence of the outflow temperature as a function of angular momentum such that the local Richardson number is not reduced below a critical minimum.

² Figure 1 of E12 suggests that both the E12 approximate analytical and full numerical solutions support a growth rate that is inversely proportional to C_d ; however, an error in the caption of Fig. 1 in E12 (C_k was varied, not C_d) does not support any conclusion on how the E12 full numerical solution growth rate depends on C_d . Although the E12 approximate equation [his (19)] suggests a growth rate inversely proportional to C_d , solutions of the full E12 numerical model, holding C_k constant and varying C_d , show that the growth rate is roughly insensitive to C_d (Emanuel 2018).

a. Control simulation

For the control simulation, C_k and C_d are set to 1×10^{-3} . Figure 1 shows a time series of the maximum tangential velocity V_m for the control experiment. The TC keeps intensifying for 72 h until it reaches an intensity of 100 m s^{-1} . A slight weakening occurs over the next several hours, and then an approximately steady state is reached after 96 h. We quantify the maximum intensity of tropical cyclones in this article by V_{max} as the maximum value of a 6-h average of the maximum tangential wind V_m from hourly output. For the control simulation, $V_{\text{max}} = 99 \text{ m s}^{-1}$.

Based on hydrostatic and gradient wind balance and the assumption of slantwise moist neutrality, the approximate diagnostic equation for the wind speed in E12 [his (13)] is

$$V_e^2 = -(T_b - T_o)M\frac{\partial s^*}{\partial M},\tag{1}$$

where V_e is the azimuthal velocity at the top of the boundary layer, T_b is the temperature at the top of the boundary layer, $T_o(M)$ is the outflow temperature, s^* is the saturation entropy, and M is the absolute angular momentum. To determine V_e from the numerical solution, we use the rhs of (1), where all terms are evaluated at the radius and height of V_m , (r_m, z_m) . For T_o , we find the temperature at the point in the upper-level outflow where the tangential velocity is zero and the radial wind speed u is largest. From Fig. 1, it is found that the diagnosed V_e is very noisy during the first 32 h; after this time there is a steady increase until 96 h, when a steady state is reached. According to the different behaviors of V_e , the intensification period for the control experiment can be divided into two phases, phase I (0-32h) and phase II (32–96 h). During phase II, V_e increases from 58 to $70 \,\mathrm{m\,s^{-1}}$ and then reaches a steady state. The tendency of V_e and V_m during phase II is qualitatively similar, as they both increase and reach a steady state at around 96h. However, the rate of intensification and maximum intensity in E12 is systematically lower than that in the numerical simulation.

At steady state, (1), together with the steady-state boundary layer entropy equation (section 2 of ER11), is essentially the maximum potential intensity (MPI) derived from the E86 analytical model. BR09a investigated the reasons for the underestimation of the model-predicted maximum intensity by the MPI and found that the neglect of unbalanced dynamics can account for the difference. We will return to this point later.

Figure 1 also shows the time series of maximum tangential wind directly integrated by the time-dependent theoretical model of E12, denoted V_{E12} ; the details of the numerical model can be found in E12. By adjusting



FIG. 1. Time series of maximum azimuthal velocity V_m (m s⁻¹) from the control simulation (black line), the velocity V_e calculated from (1) (red line), and the maximum tangential wind from an integration of the E12 model V_{E12} (purple line).

the initial distribution $s^*(M)$,³ the vortex is initialized with a maximum tangential wind of $20 \,\mathrm{m \, s^{-1}}$ [(20) in E12]. The initial size of the vortex is the same as the RE87 vortex with the tangential wind going to zero at radius $r_0 = 412.5$ km. The reference value of angular momentum $M_0 = (1/2)fr^2$. The temperatures T_o and T_b are set to 200 and 300 K, respectively; and $C_k = C_d = 1 \times 10^{-3}$. The depth of the boundary layer is 5000 m, as used in E12. In sympathy with the sentiment expressed in "The mathematical elegance of the formulation in E97 is achieved at the cost of making the physical processes of spin-up less transparent" (Montgomery and Smith 2014, p. 48)," in section 4 we transform the E12 theoretical model results from M coordinates to r-z coordinates to help clarify the intensification process embodied in the E12 model. For now, we note that the evolution of V_{E12} is very similar to V_e during phase II (Fig. 1).

b. Sensitivity simulations

To verify the generality of our results and to investigate the sensitivity of the numerical solutions to C_k and C_d , we conduct several sensitivity simulations. The values of C_k and C_d are set as 0.5×10^{-3} , 1×10^{-3} , 2×10^{-3} , and 4×10^{-3} .

Table 1 shows the maximum intensity V_{max} for all the experiments. We first observe that V_{max} in the numerical experiments is a function of C_k/C_d , as there are only

³ This is done by setting a constant that determines the size of the initial vortex, the maximum value of saturation entropy $s_l^* = 2578 \,\mathrm{J \, kg^{-1} \, K^{-1}}$, the saturation entropy of the unperturbed environment $s_e^* = 2570 \,\mathrm{J \, kg^{-1} \, K^{-1}}$, and the environment saturation entropy of air at the sea surface $s_0 = 2630 \,\mathrm{J \, kg^{-1} \, K^{-1}}$.

TABLE 1. Maximum intensity V_{max} (m s ⁻	¹) from	simulations	that
use different values for C_k (10 ⁻³)) and C_{i}	$(10^{-3}).$	

C _k		С	C_d	
	0.5	1	2	4
0.5	95	61	59	42
1	140	99	73	57
2	187	134	99	79
4	211	176	122	97

slight variations along the diagonals in the table. Furthermore, we see that when the ratio C_k/C_d is reduced from 8 to 1/8, V_{max} reduces from 211 to 42 m s⁻¹. When the drag coefficient is kept constant, the maximum intensity increases with increasing C_k . In the experiments in which C_k is held constant, the maximum intensity decreases with increasing C_d . These results are in basic agreement with the predictions of MPI. To dig a little deeper, we consider that in E86 the MPI is proportional to $(C_k/C_d)^{1/2}$, while in ER11 in which the selfstratification of TC outflow is taken into account, the MPI is modified such that

$$V_{\rm max}^2 \cong V_p^2 \left(\frac{1}{2} \frac{C_k}{C_d}\right)^{(C_k/C_d)/[2 - (C_k/C_d)]},$$
 (2)

where V_p is the MPI defined in E86. Figure 2 shows the dependence of wind speed on the ratio of exchange coefficients calculated from E86 and ER11. The results show that when C_k/C_d is less than 1, the numerical model maximum intensity is somewhat closer to the profile (2)proposed by ER11. However, when C_k/C_d is larger than 1, the relationship between maximum intensity and the ratio of C_k over C_d in the numerical simulations is closer to $(C_k/C_d)^{1/2}$. Observations suggest C_k/C_d should be between 0.6 and 0.7 (Black et al. 2007; Zhang et al. 2008) and the recent laboratory work by Haus et al. (2010) suggests a value of C_k/C_d of approximately 0.5 for winds greater than $30 \,\mathrm{m \, s^{-1}}$ up to nearly $40 \,\mathrm{m \, s^{-1}}$. Therefore, according to these observational studies, the values of C_k/C_d in nature are recognized as <1. Thus, at least within the observed range of variation, (2) is an improvement over $(C_k/C_d)^{1/2}$

According to the approximate solution for $V_m(t)$ in E12 [his (19)], the rate of intensification is proportional to $C_k V_m^2$. As discussed above in regard to Table 1, V_m is shown to be function of C_k/C_d in both numerical and theoretical results. This means that with C_k/C_d held fixed, the intensification rate should be proportional to C_k . Conversely, with C_k held fixed, the E12 approximate solution intensification rate should vary inversely with C_d (however, solutions to the unapproximated E12 model show that the intensification rate is nearly



FIG. 2. Dependence of V_{max} on the ratio of exchange coefficients from E86 (red line) and ER11 (blue line); dots are V_{max} for all simulations from Table 1.

insensitive to C_d ; see footnote 2). Figure 3a shows $V_m(t)$ from simulations with $C_k/C_d = 1$ and C_k varying from 0.5×10^{-3} to 4×10^{-3} . For the same C_k/C_d , the steady-state values of maximum intensity in the simulations are similar; however, it takes only 24 h to reach the steady state for the simulation with $C_k = 4 \times 10^{-3}$ and more than 144 h for $C_k = 0.5 \times 10^{-3}$. These results support the idea that the TC intensification rate is proportional to C_k , which is consistent with E12's theory and other previous studies (e.g., Craig and Gray 1996).

Figure 3b shows the sensitivity of the intensification rate to the variation of C_d when C_k is equal to 4×10^{-3} (the results for simulations with other values of C_k are qualitatively similar). Compared to the changes in intensification rate with changes in C_k , the intensification rate is less sensitive to changes in the drag coefficient, which is consistent with the results in Craig and Gray (1996). However, the results still suggest a proportional relationship between the intensification rate and C_d for small C_d , and then no significant difference in intensification rate is found as C_d increases beyond approximately 2×10^{-3} . These results are similar to the sensitivity of the intensification rate to changes in C_d found in Montgomery et al. (2010, their Figs. 1 and 3), which indicate the intensification rate of the vortex increased with increasing C_d until a certain threshold value is attained and then it remained relatively constant. It should be noted that, in contrast with Montgomery et al. (2010), the maximum intensity at steady state decreases with increasing C_d in our simulations.⁴ In summary, the

⁴ Bryan (2013) explains this contrast by showing a longer period of integration is needed for several of the Montgomery et al. (2010) experiments to reach a steady state.



FIG. 3. Time series of maximum azimuthal velocity V_m (m s⁻¹) from simulations with different C_k or C_d (10⁻³) for the simulations with (a) $C_k/C_d = 1$ and different values of C_k , and (b) $C_k = 4 \times 10^{-3}$ and different values of C_d .

numerical model evidence presented here and in past studies strongly suggest that the intensification rate is proportional to C_d for small values, while the prediction from the E12 model is for an intensification rate that is insensitive to C_d (Emanuel 2018).

In the following section, we investigate the reasons for the different dependence of intensification rate on C_d between the E12 theoretical model and the present numerical model.

3. Evaluation of E12

To determine why diagnosed $V_e(t)$ in (1) is not quantitatively similar to the numerical model results in phase II, we evaluate three primary components of the E12 model in the next several subsections. In the following subsections, we do not address the realism of the numerical simulations or whether the E12 model components are appropriate for natural tropical cyclones. Indeed, as discussed in BR09a and BR09b, tropical cyclones simulated with the control setup have some obviously unnatural features, such as the pseudoadiabatic assumption and small horizontal mixing length. Rather, we seek to ascertain which approximations in the E12 model are responsible for the qualitative differences in TC intensification rate between E12 and the present numerical simulation model.

a. Moist slantwise neutrality

The E12 model assumes that s^* is a function of M alone in the free atmosphere during the intensification period. In Fig. 4 we show the distribution of entropy s as formulated for pseudoadiabatic processes (Bryan 2008) and the distribution of M during the intensification in phase II for three different simulations. To a good approximation, the s and M surfaces are congruent in the eyewall in the interior free atmosphere and TC outflow. Consistent with the E12 assumption of moist slantwise neutrality, there is approximately one value of s for a given value of M in the eyewall region. During phase I, the TC intensifies while the M and s surfaces evolve from nearly orthogonal to almost congruent (not shown). The details of the intensification progress in phase I will be discussed in a follow-on paper. Based on these analyses, we conclude that an approximate condition of moist slantwise neutrality is achievable in the eyewall and outflow of a numerical simulation during the intensification in phase II. It follows that this component of E12 is probably not the source of the discrepancy between $V_m(t)$ and $V_e(t)$ in the simulations.

Figure 5 shows the s and M surfaces for the control experiment at different times during phase II. At 42 h the M surface passing through the location of V_m is at a radius of approximately 40 km; 18 h later, V_m is found at a radius of approximately 30 km. Figure 5 shows that during phase II, the s and M surfaces in the eye region are moving inward with an approximately self-similar pattern during the intensification in phase II. The M and s surfaces in the eyewall are contracting as the TC intensifies, which is similar to the eyewall frontogenesis illustrated in E97 (the predecessor of E12).

b. Gradient wind balance

BR09a showed that a major difference between the steady-state MPI of E86 and the steady states found in their numerical simulation was the presence in the model of strong non–gradient wind effects. As the E12 model is also based on the assumption of gradient wind balance, it is reasonable to look at the influence of non–gradient wind effects on the TC intensification rate. To determine the effect of the unbalanced term, we use a derivation with similar approximations to those used in BR09a and express a modified maximum wind speed V_a containing the unbalanced terms,



FIG. 4. Entropy *s* (contour interval = $10 \text{ J kg}^{-1} \text{ K}^{-1}$; green lines) and angular momentum *M* (contour interval = $0.2 \times 10^6 \text{ m}^2 \text{ s}^{-1}$; black lines) from the experiment with (a) $C_k = C_d = 0.5 \times 10^{-3}$; (b) $C_k = 4 \times 10^{-3}$, $C_d = 0.5 \times 10^{-3}$; and (c) $C_k = C_d = 4 \times 10^{-3}$. The red line is the *M* surface that passes through the location of maximum tangential wind.

$$V_{a}^{2} = -(T_{b} - T_{o})M_{b}\frac{ds}{dM} + r_{b}\eta_{b}w_{b},$$
 (3)

As it will be convenient to compare this solution to (1), we note that (3) can be expressed as follows:

$$V_a^2 = V_e^2 + r_b \eta_b w_b, \qquad (4)$$

[see also BR09a, their (20)], where the subscript *b* denotes evaluation at the top of the boundary layer, *w* is the vertical velocity component, and $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$ is the azimuthal vorticity.

where all terms are evaluated at the location of maximum tangential velocity.



FIG. 5. As in Fig. 4, but for the control experiment at (a) 42 and (b) 60 h.



FIG. 6. Time series of V_m (m s⁻¹) from the control simulation (black line), V_e calculated from (1) (red line), and V_a from (3), which includes V_e and the non–gradient wind balance term (blue line), for (a) $C_k = C_d = 1 \times 10^{-3}$, (b) $C_k = C_d = 4 \times 10^{-3}$, and (c) $C_k = 4 \times 10^{-3}$, $C_d = 0.5 \times 10^{-3}$.

Figure 6 compares the differences among V_m , V_e , and V_a in several simulations. It is clear that the evolution of $V_a(t)$ is in considerably better agreement with $V_m(t)$, which means the neglect of non–gradient wind effects in E12 may be the reason for the quantitative difference between $V_m(t)$ and $V_e(t)$. BR09a found that including unbalanced flow effects [through the second term on the rhs of (3)] in the E86 analytic model of maximum intensity

considerably improves the agreement between theory and the numerical estimates of the maximum tangential wind speed at steady state. In our study, V_a is 9 m s^{-1} less than V_m for the mature stage in the control experiment. A difference of this magnitude is seen in all the simulations (less than 10% of V_m). Figures 6b and 6c show the results for the experiments with the same C_k and different C_d $(4 \times 10^{-3} \text{ in Fig. 6b and } 0.5 \times 10^{-3} \text{ in Fig. 6c})$. When C_d is small, the non-gradient wind balance term is smaller (Bryan 2012) and the E12 model can better describe the evolution of the TC in the numerical simulations. Comparing the shape of the M surfaces in the simulation with large C_d (4 × 10⁻³; Fig. 4c) to the simulation with small C_d $(0.5 \times 10^{-3};$ Fig. 4b), the wave associated with the unbalanced flow is seen by the undulation in the M surfaces from the top of boundary layer to the tropopause.

As discussed in section 2b, the dependence of the intensification rate on C_d in the E12 model is different from the present numerical results and many other studies (e.g., Rosenthal 1971; Montgomery et al. 2010). The non–gradient wind balance term is strongly related to surface drag, which in turn is related to C_d . The present analysis suggests that the neglect of non–gradient wind effects in the diagnostic equation for wind speed of E12 [(1)] may be the reason for the different dependence of the intensification rate on C_d between the E12 model and the present numerical simulations.

c. Self-stratification of TC outflow

The self-stratification of the outflow temperature used in the E12 model is determined by small-scale turbulence that limits the Richardson number to a critical value for the onset of turbulence, which is proposed by ER11 [their (31)]:

$$\frac{\partial T_o}{\partial M} \cong -\frac{\operatorname{Ri}_c}{r_t^2} \left(\frac{dM}{ds^*}\right),\tag{5}$$

where Ri_c is the critical Richardson number and r_t is the physical radius at which the Richardson number first attains its critical value. In our simulations the region with near-critical Ri is found close to the eyewall region (not shown), which is consistent with ER11. Figure 7 shows the relationship between dT_o/dM and -dM/ds from hourly output in the control simulation during phase II. The quantity -dM/ds is calculated at the location of V_m , (r_m, z_m) . Referring to ER11, dT_o/dM is evaluated in the region $r_m + 10 \le r \le r_m + 20$ km, $12.75 \le z \le 15.25$ km for small Ri (<1). Fitting a straight line to the data between dT_o/dM and -dM/ds gives a slope of 3.1×10^{-10} m⁻², which is close to the value of Ri_c/ r_t^2 (2.7×10^{-10} m⁻²) when Ri_c = 1 and $r_t \approx 60$ km. Therefore, the self-stratification hypothesis of outflow in



FIG. 7. The relationship between dT_o/dM (K m⁻² s) and -dM/ds (K s) for hourly output from the control simulation during phase II.

the E12 model is consistent with our numerical simulations during the growth stage in phase II.

4. Other aspects of the E12 model

The foregoing results show the E12 model can qualitatively describe the evolution of $V_m(t)$ in the numerical simulations during phase II; however, the physical character of the evolution in the commonly used cylindrical coordinate system is not immediately obvious.

Figure 8 shows M(r, z, t) from the E12 theoretical model (see the appendix) in a window extending to 80 km (the full domain is 412.5 km). This figure shows that during the intensification period, the M surfaces are moving radially inward in an approximately self-similar pattern and display a type of frontogenesis, similar to the intensification features during phase II of the numerical simulations shown in Fig. 5. The derived radial and vertical velocities in the boundary layer for the E12 model are shown in Fig. 8a. Although described in detail in the appendix, a brief explanation of how the interior distribution of M(r, z, t) evolves is given here. The E12 model updates $s^*(M, t)$ in the boundary layer; (1) therefore gives V(M), which allows the calculation of r(M, t). As indicated in Fig. 8, r(M, t) shows the inward movement of large M in the boundary layer that is instantaneously communicated to the interior through the requirements of balance and moist neutrality (see the appendix). Figure 8c displays the change of M over the period of evolution shown in Fig. 8b. In the region above the boundary layer, M is conserved and therefore

 $\partial_t M = -\mathbf{u} \cdot \nabla M > 0$, which implies that the perpendicular component of the implicit secondary circulation vector, $\mathbf{u} = (u, w)$, is in the opposite direction of ∇M .

To obtain a more complete picture of the advective and turbulent processes in more realistic conditions, we return to the axisymmetric numerical simulations. Figure 9a shows the time change of M in a 12-h period, which can be compared to that of the E12 model shown in Fig. 8c. We observe a qualitative similarity, in that the maximum tendency is concentrated in the eyewall in both models over a deep layer. Major differences include the width of the region of updraft as the positive vertical motion in the E12 model (Fig. 8a) changes to negative for radii beyond roughly 200 km (not shown). Also, the inward-outward excursions of the M surfaces, and the corresponding local extrema of $\partial_t M$ evident in Fig. 9a, are absent in Fig. 8c. These local extrema are associated with non-gradient wind effects as documented by Bryan and Rotunno (2009a) for the steady-state version of the present model. The strong local positive values of $\partial_t M$ (Fig. 9a) at the top of the boundary layer support the idea that this part of the eyewall region is spun up by the vertical advection of high M from the boundary layer (Figs. 9b,c; Schmidt and Smith 2016; Kilroy et al. 2016).

Figure 10 summarizes and illustrates the qualitative similarities of the M budget in E12 and the present numerical simulation. This figure shows that while the process of inward transport of M in the boundary layer and vertical transport to the interior is explicit in the numerical simulation (cf. Fig. 9c), it is implicit in the E12 model (cf. Fig. 8a).

5. Three-dimensional simulations

Up to this point a simplified axisymmetric numerical model with simple, but unrealistic, initial conditions has been used to facilitate a comparison with the E12 theoretical model. To check whether the primary results of this study remain applicable with more complex dynamics and more realistic conditions, we have conducted an additional set of simulations using three spatial dimensions and the default physical parameterizations for CM1. Specifically, we use the microphysics scheme of Morrison et al. (2009) and a more realistic length scale in the horizontal turbulence code $(l_h = 750 \,\mathrm{m})$. Dissipative heating is included. The vertical turbulence scheme and the distribution of vertical levels are the same as before. The horizontal grid spacing is 2 km over a $400 \text{ km} \times 400 \text{ km}$ inner mesh, with increasingly stretched grid spacing beyond, with a maximum grid spacing of 16 km. The entire domain is $3000 \, \mathrm{km} \times$ 3000 km. The initial vortex is the same as before, but



FIG. 8. M(r, z) (contour interval = $0.2 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$; black lines) from the E12 timedependent model for (a) 18 and (b) 36 h. The thick arrows in (a) represent the derived inflow and updraft for E12 model. The red lines in (a) and (b) are the *M* surfaces that pass through V_m . (c) The tendency of M (m²s⁻²; color shading) averaged from 18 to 36 h. The lines in (c) are *M* surfaces passing through V_m at 18 (solid) and 36 h (dashed).

random small-amplitude temperature perturbations are added, and the initial sounding is the "moist tropical" composite from Dunion (2011). The sea surface temperature is 28°C. All simulations are integrated for 12 days.

For intensity we use the maximum value of wind speed at 10 m MSL from hourly output and then find the maximum 1-day average. The results (Table 2) have the same qualitative trends as before (Table 1), in that peak intensity increases as C_k increases or as C_d decreases. The maximum wind speeds are generally lower than before, as expected, because of the increased horizontal length scale, although perhaps in part because of a smaller airsea temperature difference. Time series of maximum wind speed (Fig. 11) show that qualitative trends in intensification rate are also the same as before (Fig. 3); that is, the intensification rate increases as C_k increases when $C_k = C_d$ (Fig. 11a) and the intensification rate increases as C_d increases when C_k is held fixed (especially for small values of C_d). These three-dimensional "full physics" simulations add confidence to the results of the more idealized simulations concerning the effects of C_d and C_k on TC intensity and intensification.

6. Summary and conclusions

In this paper we use a cloud-resolving numerical model to evaluate the time-dependent theoretical model of tropical cyclone intensification proposed by E12. Applying the diagnostic relation [(1)] from E12 to the present numerical simulations indicates a very noisy $V_e(t)$ during the initial adjustment of the vortex to a state of moist neutrality along angular momentum surfaces (phase I). After this adjustment is complete, $V_e(t)$ shows a more regular increase (phase II) that is qualitatively similar to the numerical simulations but not quantitatively. The results of the sensitivity experiments for different drag (C_d) and entropy (C_k) surface exchange coefficients show that the maximum tangential wind speed in the numerical experiments is a function of C_k/C_d and with a



FIG. 9. Analysis of the inward transport of the angular momentum surfaces through (a) the time change (m² s⁻², averaged from 48 to 60 h), (b) selected *M* surfaces (contour interval = 0.2×10^{-6} m² s⁻¹) and the radial–vertical wind vectors in the box shown in (a), and (c) the profiles of vertical advection, horizontal advection, and diffusion of *M* (m² s⁻²) at the location of its maximum time change. The lines in (a) are *M* surfaces passing through V_m at 48 (solid) and 60 h (dashed), and the position of V_m is indicated by the dot in (b).

functional dependence that is closer to that found by ER11 for the observationally relevant range in which $C_k/C_d < 1$. The intensification rate in numerical simulations is proportional to C_k , which is consistent with E12's theory. However, the intensification rate is also proportional to C_d within a certain range in our simulations, while the E12 model is essentially insensitive to C_d . These results hold for both axisymmetric and three-dimensional numerical simulations, using both simplified and realistic physical parameterizations and initial conditions.

We analyze the present numerical simulations to evaluate the assumptions used in the E12 model to find the reason for the difference in results between the E12 model and our numerical model. It is found that during phase I, the TC intensifies while the angular momentum (M) and entropy (s) surfaces evolve from nearly orthogonal to almost congruent, which is not consistent with slantwise moist neutrality. The details of the intensification progress in phase I will be discussed in another paper. During phase II, the M and s surfaces are congruent as the TC intensifies, which means the assumption of slantwise moist neutrality in E12 is valid in the intensification process of phase II.

The effect of non-gradient wind balance on TC intensification and the mature stage in eyewall region has been recognized by several studies (e.g., Smith et al. 2008; BR09a). In the present analysis, we find that the inclusion of non-gradient wind effects in the theoretical framework of E12 produces an intensification rate that is quantitatively similar to the numerical model results. As the non-gradient wind term is closely related to the drag efficient C_d , the neglect of non-gradient wind effects in E12 seems likely to be the reason for the different dependence on C_d of the intensification rate between E12 and the present numerical model.

The self-stratification of the outflow temperature used in the E12 model is also evaluated in our numerical simulation. We found that the region with nearcritical Ri is close to the eyewall region; fitting a straight line to the data between dT_o/dM and -dM/ds gives a slope of 3.1×10^{-10} m⁻², which is close to the value of Ri_c/ r_t^2 (2.7×10^{-10} m⁻²) when Ri_c = 1 and $r_t \approx 60$ km. These results indicate that the hypothesis of the outflow self-stratification determined by small-scale turbulence that limits the Richardson number to a critical value for the onset of turbulence is appropriate in our numerical simulation during the intensification process of phase II.

Other aspects of the E12 model in the context of recent research on TC intensification are discussed. When transformed to cylindrical coordinates, the qualitative similarity of the E12 model to the present model becomes apparent. Noted differences are that the width of the updraft/eyewall region is much smaller and



FIG. 10. Schematic diagram of the angular momentum (M) budget of (a) the present numerical model (cf. Fig. 9) and (b) the E12 model (cf. Fig. 8). In the boundary layer, both models exhibit the inward transport of M and the loss of M to the lower surface. At a fixed radius in the numerical model in (a), M has a local maximum at a level lower than the level where u = 0, so there is both inward and upward transport between these two levels. In the E12 model in (b), the upward transport at the top of its boundary layer is implicit. Above the boundary layer, the numerical model has a radial–vertical velocity such that there is a component in the opposite direction of ∇M , while this velocity is implicit in the E12 model.

non–gradient wind effects are apparent in the cloud model. The present analysis of the cloud model angular momentum budget supports the idea that vertical advection from the boundary layer plays a role in eyewall spinup at the top of the boundary layer (e.g., Schmidt and Smith 2016; Kilroy et al. 2016).

The E12 model for TC intensification is the only theoretical model with an approximate analytical solution at present. Compared to a nonhydrostatic cloud model, the theoretical model possesses the chief virtue of transparency. On the other hand, transparency usually comes at the expense of accuracy. The purpose of this study is to examine the trade-off between transparency and accuracy in the E12 model of TC intensification. It is hoped that identified strengths and weaknesses can guide

TABLE 2. Maximum wi	nd speed $(m s^{-1})$	at 10 m MSL (a	iveraged
over 1 day, using hourly	output) from 3D	full-physics sin	nulations

	$C_d (\times 10^3)$			
C_k (×10 ³)	0.5	1	2	4
0.5	83	70	54	40
1	105	85	66	50
2	118	104	80	60
4	131	113	89	70



FIG. 11. As in Fig. 3, but for 3D full-physics simulations.

future attempts at the construction of a more accurate, yet sufficiently transparent, theoretical model.

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APPENDIX

Transformation from *M* Coordinates to *r*-*z* Coordinates

According to E12's theoretical model, we can obtain values of s(M), $T_o(M)$, V(M), and radius $r_b(M)$ at the top of the boundary layer. The complete two-dimensional structure of the theoretical TC can be obtained by evaluating quantities along M surfaces. The equation for r_b^2 is given by (4) of E12. For convenience, r(M, T) can be calculated as follows:

$$r^{2}(M,T) = r_{b}^{2}(M) \frac{T_{b} - T_{o}(M)}{T - T_{o}(M)}.$$
 (A1)

We then obtain the tangential velocity V(M, T) from the absolute angular momentum—that is, V = M/r - (1/2)fr—and finally we interpolate V(M, T) to r-Tcoordinates to obtain V(r, T).

To transform all the variables from r-T to r-z coordinates, the relation between z and T needs to be found. As a matter of convenience, the height of the M surfaces may be obtained by noting that the constancy of saturated equivalent potential temperature θ_e^* along M surfaces implies that the saturated moist static energy, defined as

$$h^* = C_p T + gz + Lq^*, \tag{A2}$$

is approximately constant along such surfaces. The height z is obtained from (A2):

$$z = \frac{C_p}{g}(T_b - T) + \frac{L}{g}[q_b^*(\pi_b) - q^*(\pi, T)], \quad (A3)$$

where $\pi = (p/p_0)^{R_d/C_p}$ is nondimensional pressure; p_0 is a reference pressure; C_p and R_d are the specific heat and gas constants of dry air, respectively; q^* is the saturation mixing ratio; and L is the latent heat of vaporization. Since q^* is a function of temperature and pressure, which is unknown, we derive it in order to infer the height of the surfaces from (A3).

The saturation mixing ratio at the top of the boundary layer at constant temperature may be expressed as

$$q_b^* \approx q_{ab}^* \left[1 - \frac{C_p}{R_d} \ln \pi_b \left(r \right) \right], \tag{A4}$$

where the subscript *a* denotes the ambient variable. The gradient wind relationship can be written as

$$M^{2} = r^{3} \left(C_{p} T_{b} \frac{\partial \ln \pi}{\partial r} + \frac{1}{4} f^{2} r \right).$$
 (A5)

This may be directly integrated once in *r* and then the boundary layer pressure π_b can be obtained as follows:

$$\pi_{b}(r) = \pi_{a} + \exp\left\{\frac{1}{C_{p}T_{b}}\left[\int_{r_{0}}^{r}\frac{M^{2}}{r^{3}}dr - \frac{1}{8}f^{2}(r^{2} - r_{0}^{2})\right]\right\};$$
(A6)

upon substitution of (A6) into (A4), we obtain $q_b^*(r)$.

The saturation mixing ratio is found from the Clausius– Clapeyron equation:

$$q^* = q_b^* \left(\frac{\pi_b}{\pi}\right)^{C_p/R_d} \exp\left[\frac{L}{R_v} \left(\frac{1}{T_b} - \frac{1}{T}\right)\right], \quad (A7)$$

where R_v is the gas constant for water vapor. According to the definition of saturation entropy, the value of π can be expressed as

$$\ln \pi = \frac{s^*}{C_p} + \ln T + \frac{Lq^*}{C_p T}.$$
 (A8)

Combining (A7) and (A8), the saturation mixing ratio for each T, $q^*(T, r)$, can be calculated.

Substituting $q_b^*(r)$ and $q^*(T, r)$ into (A3), we obtain the desired relation between z and T. Finally, we interpolate V(r, T) to V(r, z). Other variables can be transformed from M coordinates to r-z coordinates by the same method.

To derive the radial velocity u in the boundary layer, we consider the M equation in cylindrical coordinates,

$$\frac{\partial M}{\partial t} + u \frac{\partial M}{\partial r} = -\frac{\partial \tau_M}{\partial z},\tag{A9}$$

where τ_M is the vertical flux of M, which is expressed using the standard aerodynamic formula, $\tau_M = -C_d |V| r V$.

The anelastic form of the mass continuity equation can be written as

$$\frac{\partial}{\partial r}(\rho r u) + \frac{\partial}{\partial z}(\rho r w) = 0, \qquad (A10)$$

where ρ is the density and w is the vertical velocity. A mass streamfunction ψ may be defined by virtue of (A10), such that

$$\rho r u = -\frac{\partial \psi}{\partial z},\tag{A11}$$

and substituting (A11) into (A9) results in

$$\frac{\partial \psi}{\partial z} \frac{\partial M}{\partial r} = \rho r \left(\frac{\partial M}{\partial t} + \frac{\partial \tau_M}{\partial z} \right).$$
(A12)

Since density varies only slightly within the boundary layer, (A12) may be integrated in *z* to yield

$$\psi = \rho_0 r \left(h \frac{\partial M}{\partial t} \Big|_{z=h} + \tau_M \Big|_{z=h} - \tau_M \Big|_{z=0} \right) / \frac{\partial M}{\partial r} \Big|_{z=h}, \quad (A13)$$

where ρ_0 is a mean density in the boundary layer, *h* is the depth of boundary layer, and the boundary condition $\psi = 0$ at z = 0 was used.

The vertical mean radial velocity in the boundary layer may be found by integrating (A11) across the depth of the layer so that

$$r\overline{u} = -\frac{1}{\rho_0 h}\psi. \tag{A14}$$

The vertical velocity is related to the streamfunction and can be expressed as

$$\rho r w = \frac{\partial \psi}{\partial r}.$$
 (A15)

Substituting (A13) into (A14) and (A15), we can obtain the vertical mean radial velocity \overline{u} and the vertical velocity w.

REFERENCES

- Black, P. G., and Coauthors, 2007: Air-sea exchange in hurricanes: Synthesis of observations from the Coupled Boundary Layer Air-Sea Transfer experiment. *Bull. Amer. Meteor. Soc.*, 88, 357–374, https://doi.org/10.1175/BAMS-88-3-357.
- Bryan, G. H., 2008: On the computation of pseudoadiabatic entropy and equivalent potential temperature. *Mon. Wea. Rev.*, 136, 5239–5245, https://doi.org/10.1175/2008MWR2593.1.
- —, 2012: Effects of surface exchange coefficients and turbulence length scales on the intensity and structure of numerically simulated hurricanes. *Mon. Wea. Rev.*, 140, 1125–1143, https:// doi.org/10.1175/MWR-D-11-00231.1.
- —, 2013: Comments on 'Sensitivity of tropical-cyclone models to the surface drag coefficient.' *Quart. J. Roy. Meteor. Soc.*, **139**, 1957–1960, https://doi.org/1002/qj.2066.
- —, and R. Rotunno, 2009a: Evaluation of an analytical model for the maximum intensity of tropical cyclones. J. Atmos. Sci., 66, 3042–3060, https://doi.org/10.1175/2009JAS3038.1.
- , and —, 2009b: The maximum intensity of tropical cyclones in axisymmetry numerical model simulations. *Mon. Wea. Rev.*, **137**, 1770–1789, https://doi.org/10.1175/2008MWR2709.1.
- Craig, G. C., and S. L. Gray, 1996: CISK or WISHE as the mechanism for tropical cyclone intensification. J. Atmos. Sci., 53, 3528–3540, https://doi.org/10.1175/1520-0469(1996)053<3528: COWATM>2.0.CO;2.
- Dunion, J. P., 2011: Rewriting the climatology of the tropical North Atlantic and Caribbean Sea atmosphere. J. Climate, 24, 893–908, https://doi.org/10.1175/2010JCLI3496.1.
- Emanuel, K. A., 1986: An air-sea interaction theory for tropical cyclones. Part I: Steady-state maintenance. J. Atmos. Sci., 43, 585–604, https://doi.org/10.1175/1520-0469(1986)043<0585: AASITF>2.0.CO:2.
- —, 1997: Some aspects of hurricane inner-core dynamics and energetics. J. Atmos. Sci., 54, 1014–1026, https://doi.org/ 10.1175/1520-0469(1997)054<1014:SAOHIC>2.0.CO;2.
- —, 2012: Self-stratification of tropical cyclone outflow. Part II: Implications for storm intensification. J. Atmos. Sci., 69, 988–996, https://doi.org/10.1175/JAS-D-11-0177.1.
- —, 2018: Corrigendum. J. Atmos. Sci., https://doi.org/10.1175/ JAS-D-18-0047.1, in press.
- , and R. Rotunno, 2011: Self-stratification of tropical cyclone outflow. Part I: Implications for storm structure. *J. Atmos. Sci.*, 68, 2236–2249, https://doi.org/10.1175/JAS-D-10-05024.1.
- Haus, B. K., D. Jeong, M. A. Donelan, J. A. Zhang, and I. Savelyev, 2010: Relative rates of sea-air heat transfer and frictional drag

in very high winds. *Geophys. Res. Lett.*, **37**, L07802, https://doi.org/10.1029/2009GL042206.

- Kilroy, G., R. K. Smith, and M. T. Montgomery, 2016: Why do model tropical cyclones grow progressively in size and decay in intensity after reaching maturity? J. Atmos. Sci., 73, 487– 503, https://doi.org/10.1175/JAS-D-15-0157.1.
- Montgomery, M. T., and R. K. Smith, 2014: Paradigms for tropical cyclone intensification. *Aust. Meteor. Oceanogr. J.*, 64, 37–66.
 —, and —, 2017: Recent developments in the fluid dynamics of tropical cyclones. *Ann. Rev. Fluid Dyn.*, 49, 541–574, https://doi.org/10.1146/annurev-fluid-010816-060022.
- —, R. Smith, and N. Van Sang, 2010: Sensitivity of tropical cyclone models to the surface drag coefficient. *Quart. J. Roy. Meteor. Soc.*, **136**, 1945–1953, https://doi.org/10.1002/qj.702.
- Morrison, H., G. Thompson, and V. Tatarskii, 2009: Impact of cloud microphysics on the development of trailing stratiform precipitation in a simulated squall line: Comparison of oneand two-moment schemes. *Mon. Wea. Rev.*, **137**, 991–1007, https://doi.org/10.1175/2008MWR2556.1.

- Rosenthal, S. L., 1971: The response of a tropical cyclone model to variations in boundary layer parameters, initial conditions, lateral boundary conditions and domain size. *Mon. Wea. Rev.*, 99, 767–777, https://doi.org/10.1175/1520-0493(1971)099<0767: TROATC>2.3.CO;2.
- Rotunno, R., and K. Emanuel, 1987: An air–sea interaction theory for tropical cyclones. Part II: Evolutionary study using a nonhydrostatic axisymmetric numerical model. J. Atmos. Sci., 44, 542–561, https:// doi.org/10.1175/1520-0469(1987)044<0542:AAITFT>2.0.CO:2.
- Schmidt, C., and R. K. Smith, 2016: Tropical cyclone evolution in a minimal axisymmetric model revisited. *Quart. J. Roy. Meteor.* Soc., 142, 1505–1516, https://doi.org/10.1002/qj.2753.
- Smith, R. A., M. T. Montgomery, and S. Vogl, 2008: A critique of Emanuel's hurricane model and potential intensity theory. *Quart.* J. Roy. Meteor. Soc., 134, 551–561, https://doi.org/10.1002/qj.241.
- Zhang, J. A., P. G. Black, J. R. French, and W. M. Drennan, 2008: First direct measurements of enthalpy flux in the hurricane boundary layer: The CBLAST results. *Geophys. Res. Lett.*, 35, L14813, https://doi.org/10.1029/2008GL034374.