## Numerical Simulations of Two-Layer Flow past Topography. Part II: Lee Vortices

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#### ABSTRACT

This study considers a two-layer fluid with constant density in each layer connected by a layer of continuously varying density for flows past topography in which hydraulic jumps with lee vortices are expected based on shallow-water theory. Numerical integrations of the Navier–Stokes equations at a Reynolds number high enough for a direct numerical simulation of turbulent flow allow an examination of the internal mechanics of the turbulent leeside hydraulic jump and how this mechanics is related to lee vortices. Analysis of the statistically steady state shows that the original source of lee-vortex vertical vorticity is through the leeside descent of baroclinically produced spanwise vorticity associated with the hydraulic jump. This spanwise vorticity is tilted to the vertical at the spanwise extremities of the leeside hydraulic jump. Turbulent energy dissipation in flow through the hydraulic jump allows this leeside vertical vorticity to diffuse and extend downstream. The present simulations also suggest a geometrical interpretation of lee-vortex potentialvorticity creation, a concept central to interpretations of lee vortices based on the shallow-water equations.

#### 1. Introduction

This two-part study is motivated by the desire to develop a better understanding of the physical relation between the shallow-water equations (SWE) and the Navier-Stokes equations (NSE) in their representations of two-layer stratified flow past topography. In a fluid composed of two layers of different, but constant, densities, all the vorticity is concentrated at the interface between the two layers; this view of vorticity is complicated in cases where the generally horizontally oriented density interface forms a discontinuity in its vertical displacement (i.e., a "hydraulic jump"), such as can occur in the SWE (Rotunno and Smolarkiewicz 1995). In this two-part study, we examine numerical solutions of the three-dimensional NSE for a two-layer fluid, with a thin connecting layer of continuous density variation, flowing past topography for situations in which solutions to the SWE indicate hydraulic jumps.

Hydraulic jumps play a critical role in solutions to the SWE exhibiting lee vortices (counterrotating eddies immediately downstream of a mountain barrier), as shown in Fig. 1 (from Epifanio 2014). These solutions are classified into two distinct types: one where the interfacial height remains above the obstacle (Figs. 1a,b), and the other, where the density interface is pierced by

the obstacle (Figs. 1c,d). In the former case, two oppositely signed bands of vertical vorticity extend downstream from the spanwise termination points of a leeside hydraulic jump (Fig. 1b); in the latter case, these bands extend downstream from hydraulic jumps on the flanks of the obstacle where it pierces the interfacial layer (Fig. 1d). Owing to the added complexity of the numerical solution to the SWE for the case where the obstacle pierces the interface (e.g., Schär and Smith 1993, their appendix B), the present two-part study is restricted to the case where the interface remains above the obstacle as in Figs. 1a and 1b. The pierced-layer case is briefly discussed in section 3c.

In the representation of two-layer flow in the SWE, the vertical vorticity  $\zeta$  is that of the lower layer (of thickness *h*) and is only nonzero if the potential vorticity  $\zeta/h$  is nonzero, which, in the flows over frictionless topography under consideration, can only occur when there is internal energy dissipation. In the case of the interfacial layer remaining above the obstacle, the solution to SWE has  $\zeta/h = 0$  until the flow passes through the horizontal discontinuity in the interfacial layer height (the hydraulic jump); energy dissipation in the flow through the hydraulic jump produces a dipole of  $\zeta/h$  and thus (because h > 0) oppositely signed bands of  $\zeta$  extending downstream from the spanwise extremities of the hydraulic jump (Fig. 1b). Hence the conclusion from the SWE is that lee vortices are produced by

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FIG. 1. Lee vortices in free-slip shallow-water flow. Shown are (a) streamlines and (b) vertical vorticity for a case in which the fluid layer completely covers the obstacle, with supercritical flow along the lee slope. (c),(d) As in (a) and (b), but for a case in which the obstacle pierces the fluid surface, causing the flow to split on the upstream face. Heavy solid lines in all panels show the positions of hydraulic jumps. The blank region in (c) shows the part of the obstacle above water level. After Schär and Smith (1993). [Figure and caption taken from Epifanio (2014).]

dissipative processes associated with the hydraulic jump (Schär and Smith 1993, section 2d).

The critical role of hydraulic jumps in the SWE for lee-vortex formation motivates the present study. Rotunno and Bryan (2018, hereafter Part I) examined the flow past a two-dimensional (no spanwise variation) obstacle for cases in which the SWE produce a leeside hydraulic jump. The solutions to the NSE for this flow indicate the hydraulic jump is a leeside wave in the thin density-transition layer that overturns backward (against the ambient flow) toward the obstacle leading to a time- and spanwise-mean rotary motion and turbulence; the spanwise vorticity for the mean rotary motion is supplied by baroclinic vorticity production in the thin interfacial layer as it descends on the leeside of the obstacle and flows into the wave (Fig. 11 of Part I). Here in Part II we examine direct numerical solutions of the NSE to study the same two-layer fluid as in Part I, but for flow past a three-dimensional obstacle (limited spanwise extent). These solutions allow an analysis of the origin of lee-vortex vorticity and its relation to the continuous-fluid version of the hydraulic jump in a fully resolved turbulent flow.

In section 2a, we briefly review the experimental setup for the numerical solutions of the NSE which, except for the three-dimensional obstacle, is identical to that in Part I (where the differences between hydraulic jumps in the traditional water-air system and the atmospheric case are discussed). In section 2b we present the results. Our analysis of the lee vortices found in the NSE in terms of vorticity and Ertel potential vorticity is given in section 3 and concluding remarks in section 4.

# 2. Two-layer flow past a submerged obstacle in the NSE

## a. Numerical setup

The numerical setup for the present numerical experiment is described in Part I; the only changes here are the spanwise-varying topography and a spanwise doubling of the integration domain keeping the same grid resolution ( $\Delta x = \Delta y = \Delta z = 20 \text{ m}$ ). For this study, the obstacle height H(x, y) is specified following (7) and (8) of Epifanio and Rotunno (2005) with parameters  $h_0 =$  $1000 \,\mathrm{m}$  (maximum obstacle height),  $L = 500 \,\mathrm{m}$  (a horizontal scale) and  $\beta = 3$  (which determines the spanwise extent of the obstacle;  $\beta = 1$  produces a circular obstacle). Figure 2 shows the obstacle and integration domain in a three-dimensional perspective. As in Part I, the domain in the along-flow (x) direction extends 15.36 km with "open" boundary conditions applied at both ends, while the domain in the vertical (z) direction extends from the height of the zero-stress topography z = H(x, y) to z = 3.84 km where there is a zero-stress, rigid lid. The domain in the spanwise (y) direction extends 7.68 km with periodicity assumed at the lateral boundaries.

The initial potential-temperature profile is given by (9) of Part I and is shown in the plane y = 3.84 km in Fig. 2; the profile is similar to the two-layer idealization of the SWE except for the finite-thickness transition layer connecting the lower-fluid to the upper-fluid constant potential temperatures. The initial velocity field (u, v, w) = (U, 0, 0), in the Cartesian coordinates (x, y, z), quickly adjusts to potential flow past the obstacle. As discussed in Part I, the initial-condition parameters for the potential-temperature profile, the mountain height and the upstream velocity,  $U = 5 \text{ m s}^{-1}$ , are chosen based on shallow-water-theory predictions for a stationary leeside hydraulic jump.

As described in Part I, the numerical simulations performed for this study are in the category of direct numerical simulations (DNSs) in which no subgrid turbulence parameterization is used. Using a constant kinematic viscosity and thermometric conductivity ( $\nu = \kappa = 1 \text{ m}^2 \text{ s}^{-1}$ ), one can estimate the Reynolds number ( $Uh_0/\nu$ ) to be approximately 5000—small compared to the real atmosphere, but large enough to capture turbulence motions without approximation. A spectrum analysis (not shown) of the vertical velocity in the zone



FIG. 2. Domain, topography, and initial conditions (at y = +3.84 km) for the present numerical solutions. Isolines of potential temperature  $\theta$  in intervals of 1 K starting with 289 K (green contour lines, with the 293 K contour in red) and vectors representing the initial velocity (u, v, w) = (5, 0, 0) m s<sup>-1</sup> are shown. The domain extends 15.36 km in the along-flow (x), 7.68 km in the spanwise (y), and 3.84 km in the vertical (z) direction.

of high turbulent kinetic energy (Fig. 7b of Part I) indicates that the energy in the largest turbulent scale is at a wavelength of approximately 1000 m and that the spectrum roughly follows a -5/3 decrease with scale down to the smallest resolved turbulent scale which is approximately 100 m. The ratio of the grid length (20 m) to the largest turbulent scale is therefore 0.02, which is within the recommended upper limit of  $15 \times$  (Reynolds number)<sup>-3/4</sup>  $\approx$  0.025 for accurate DNS of free-shear layers (Moin and Mahesh 1998, p. 554).

## b. Results

Figure 3 shows a time sequence of the flow (u, w)and potential temperature  $\theta$  in a vertical (x-z) cross section taken along the flow on the obstacle centerline (y = 0). Figure 4 shows the same fields except in a horizontal (x-y) cross section at z = 0.4 km. Figures 3a and 4a show the flow at time t = 1 min after initialization so that the rapid adjustment to potential flow around the obstacle is in evidence. Consistent with shallow-water theory and with the flow past a two-dimensional obstacle (Fig. 3 of Part I), the flow in this regime exhibits a leeside steepening of the isentropes (Fig. 3b) that continues to isentrope overturning and local flow reversal (Figs. 3c, 4b). As in the 2D case in Fig. 3 of Part I, warm air is extruded downward and mixes with the cooler lowerlevel air; however, in the present case, the initial warmair bubble is much smaller and mixes more rapidly. The flow adjusts to a sequence of such episodes of warm-air extrusion into, and mixing with, the lower-layer cool air as illustrated in Figs. 3d and 3e.

Figures 5 and 6 show a windowed-in view of the centerline x-z cross sections of  $\theta$ , flow vectors, and spanwise



FIG. 3. Isolines of the potential temperature  $\theta$  along the flow through the obstacle centerline (y = 0) in intervals of 1 K starting with 289 K (green contour lines, with the 293 K contour in red) and vectors representing (u, w) for t = (a) 1, (b) 8, (c) 11, (d) 50, and (e) 60 min.

vorticity  $\eta = \partial_z u - \partial_x w$  (top panels) along with *x*-*y* sections of  $\theta$ , flow vectors, and vertical vorticity  $\zeta = \partial_x v - \partial_y u$  at z = 0.4 km.

Figure 5a shows a snapshot of the flow at t = 11 min; the flow shown in the upper panel is qualitatively similar to that expected from the simulations past a 2D obstacle shown in Fig. 5 of Part I in which  $\eta < 0$ , baroclinically generated along the descending isentropes, induces a rotary motion with reversed flow. The reversed flow splits the layer vertically and elevates the upper isentropes which then baroclinically produce  $\eta > 0$  above the reversed flow. The bottom panel of Fig. 5a shows the isentropes that extend below z = 0.4 km and the flow that advects  $\theta$  into two local maxima collocated with the two members of the dipole in  $\zeta$ . We note that at this time there is no turbulence in evidence and that, therefore, the flow that results from this initial-value problem produces lee vortices in the absence of turbulent dissipation. Figure 5b shows that turbulent





FIG. 4. Vectors representing (u, v) at z = 0.4 km for t = (a) 1 and (b) 11 min along with vertical vorticity  $\zeta$  (color shading) and isolines of potential temperature  $\theta$  as in Fig. 3.

motions follow subsequently as the extruded warm air becomes cut off from its leeside source (top panel) and the initial  $\zeta/\theta$  anomalies drift downstream (bottom panel).

Figures 6a and 6b show the next episode of extruded warm air with baroclinic production of  $\eta < 0$  and subsequent mixing (top panels) and the associated formation and downstream drift of a  $\zeta$  dipole (bottom panels). At this stage the wake turbulence from the previous episode leads to a much less orderly flow. Episodes like those illustrated in the previous two figures continue throughout the simulation. After approximately 60 min, the flow settles into a fully developed statistically steady turbulent flow shown in Fig. 7a. Figure 7b shows the turbulent kinetic energy (tke) based on the 60–120-min average of 1-min output data.<sup>1</sup> The top panels in Fig. 7 showing the centerline flow and tke are qualitatively consistent with Fig. 7 of Part I in the 2D-obstacle case. The bottom panels of Fig. 7 show that the most intense turbulence is well correlated with the time-averaged vertical vorticity downstream of the location where tke and  $\zeta$  are generated (near  $x \approx -1.0$  km).

## 3. Analysis

## a. Vorticity

Analysis of lee-vortex formation in a continuously stratified fluid by Schär and Durran (1997, p. 544) confirms that baroclinic generation and tilting as proposed by Smolarkiewicz and Rotunno (1989) and Rotunno and Smolarkiewicz (1991) account for these vortices during the start-up phase, which lasts for approximately one advective time scale  $(4L/U \simeq 6.67 \text{ min in the present})$ case). This first period is called the "inviscid phase" (analogous to the results shown in Figs. 3a-c and 5a) while the second and much longer period analyzed by Schär and Durran (1997) is called the "dissipative phase" (analogous to the present results shown in Figs. 3d and 3e and Figs. 6 and 7). In this latter phase, Schär and Durran (1997) analyze the lee vortices in terms of the Ertel potential vorticity, which in its Boussinesq form is

$$PV = \boldsymbol{\omega} \cdot \nabla \boldsymbol{\theta} \tag{1}$$

where  $\boldsymbol{\omega} = (\xi, \eta, \zeta)$  is the vorticity vector. As argued in Rotunno et al. (1999), analysis of the lee vortices solely in terms of PV leaves unanswered the question of how  $\boldsymbol{\omega}$  is created in the steady-state flow as fluid parcels traverse the obstacle from upstream where  $\boldsymbol{\omega} = 0$ . This section provides an answer to this question for the case at hand based on the present numerical simulations.

To isolate the essential source(s) of the vertical vorticity of the lee eddies, it is convenient to work with the flux form of the vector vorticity equation (section 5 of Haynes and McIntyre 1987) which has the property that the flux of vorticity in its own direction is zero. Consequently, the vertical flux of vertical vorticity is identically zero and the present analysis reduces to

<sup>&</sup>lt;sup>1</sup> Clearly a longer averaging period and/or more frequent outputs would be required to produce more spatially homogeneous distributions of these fields.



FIG. 5. (top) Windowed-in view of centerline  $\theta$  and flow vectors as in Fig. 3 with the addition of the spanwise vorticity  $\eta$  in color shading and (bottom)  $\theta$  (green contours), flow vectors, and vertical vorticity  $\zeta$  at z = 0.4 km for t = (a) 11 and (b) 17 min. The horizontal line in the top panels shows the z = 0.4 km level displayed in the bottom panels.

determining the horizontal fluxes of vertical vorticity into and out of an area surrounding a lee vortex.

Following Rotunno et al. [1999, their (2.14)], the timeaveraged, flux-form equation for the vertical component of vorticity in a statistically steady flow is

$$\partial_x Z_{zx} + \partial_y Z_{zy} = 0 \tag{2}$$

where  $(Z_{zx}, Z_{zy}) = (\overline{u\zeta} - \overline{w\xi} - \nu\nabla^2 \overline{v}, \overline{v\zeta} - \overline{w\eta} + \nu\nabla^2 \overline{u})$ and the overbar indicates the (60–120 min) time average. The viscous terms are small and will be neglected in the following. Because there is no net vertical vorticity (i.e., circulation) created on a level circuit enclosing the entire disturbed flow, we focus here on the flow in the half-space y < 0.

The flow in the square shown in the bottom panel of Fig. 7a is chosen for detailed study since  $\overline{\zeta} = 0$  on its upstream side at x = -1.3 km and flank side at y = -1.5 km, and therefore, all the vertical vorticity in the square must flow through its borders at y = 0 and x = 0.2 km. The vertical vorticity flux vector ( $Z_{zx}, Z_{zy}$ ) in this square is shown in Fig. 8a. This figure shows a vertical

vorticity flux into the square along y = 0 and out of the square along x = 0.2 km. Integrating (2) over the area of the square at z = 0.4 km gives

$$\int_{y=-1.5}^{y=0} Z_{zx}|_{x=0.2} dy = -\int_{x=-1.3}^{x=0.2} Z_{zy}|_{y=0} dx$$
(3)

since the fluxes are zero on the other two sides and the limits are given in kilometers. Numerical evaluation of the rhs yields a value of  $28.65 \text{ m}^2 \text{ s}^{-2}$  while the integral on the lhs yields  $32.86 \text{ m}^2 \text{ s}^{-2}$ . Recognizing the noisy fields near the right side of the square, we consider a square with a variable right-side position  $\hat{x}$  so that integration of (2) over the area of the square gives

$$\int_{y=-1.5}^{y=0} Z_{zx}|_{x=\hat{x}} dy = -\int_{x=-1.3}^{x=\hat{x}} Z_{zy}|_{y=0} dx ; \qquad (4)$$

the average of (4) from  $\hat{x} = -0.2$  to 0.2 gives

$$\int_{y=-1.5}^{y=0} \langle Z_{zx} \rangle \, dy = -\left\langle \int_{x=-1.3}^{x=\hat{x}} Z_{zy} \big|_{y=0} \, dx \right\rangle \tag{5}$$



FIG. 6. As in Fig. 5, but for t = (a) 23 and (b) 29 min.

where the angle brackets indicate the average. With this adjustment, the integral on the lhs of (5) then yields a value of  $28.35 \text{ m}^2 \text{ s}^{-2}$  while the integral on the rhs is  $29.21 \text{ m}^2 \text{ s}^{-2}$  which indicates that the equality (5) holds to an accuracy of ~3%.

To make progress in the interpretation of (2), we decompose the flow into time-mean and fluctuating components given by  $\chi = \overline{\chi} + \chi'$  where  $\overline{\chi'} = 0$ . With this decomposition,  $(Z_{zx}, Z_{zy}) = (\overline{Z}_{zx}, \overline{Z}_{zy}) + (Z'_{zx}, Z'_{zy})$ , where  $(\overline{Z}_{zx}, \overline{Z}_{zy}) = (\overline{u}\overline{\zeta} - \overline{w}\overline{\xi}, \overline{v}\overline{\zeta} - \overline{w}\overline{\eta})$  and  $(Z'_{zx}, Z'_{zy}) = (\overline{u'\zeta'} - w'\xi', \overline{v'\zeta'} - \overline{w'\eta'})$ . With these definitions, (2) becomes

$$\partial_x \overline{Z}_{zx} + \partial_y \overline{Z}_{zy} = -(\partial_x Z'_{xz} + \partial_y Z'_{zy}). \tag{6}$$

Note that the rhs is equivalent to  $\partial_x F_y - \partial_y F_x$  where  $F_i = -\partial \overline{u'_i u'_j} / \partial x_j$  is (minus) the divergence of the Reynolds stress.<sup>2</sup>

Our analysis of (6) (not shown) indicates that the Reynolds stress terms are relatively small at the level of

the square (z = 0.4 km, Fig. 7a) as well as at the level of maximum tke (z = 0.2 km, Fig. 7b) and the main balance is between the terms on the lhs. This conclusion is consistent with the finding of Epifanio and Qian (2008, their section 5a) in a large-eddy simulation of a largeamplitude mountain wave that the direct effects of the Reynolds stress on the mean-flow modification are negligible. Furthermore, our analysis indicates that ( $\overline{Z}_{zx}$ ,  $\overline{Z}_{zy}$ ) can be approximated by ( $\overline{u}\,\overline{\zeta}, -\overline{w}\,\overline{\eta}$ ); Fig. 8b shows close quantitative agreement of the exact and approximate forms of the vertical-vorticity flux and motivates the evaluation of the following approximate form of (3),

$$\int_{y=-1.5}^{y=0} \langle \overline{u}\,\overline{\zeta}\rangle\,dy \approx \int_{x=-1.3}^{x=0.2} (\overline{w}\,\overline{\eta})\big|_{y=0}\,dx.$$
 (7)

Figure 9 shows the integrands in (5) and (7); the cross-stream flux of vertical vorticity (Fig. 9a) is closely followed by its approximation while the down-stream flux is more variable (Fig. 9b). Numerical evaluation of (7) yields a rhs value of  $33.66 \text{ m}^2 \text{ s}^{-2}$  while the lhs yields  $30.25 \text{ m}^2 \text{ s}^{-2}$  which indicates that (7) holds to an accuracy of ~10%. From the foregoing analysis we

<sup>&</sup>lt;sup>2</sup> The indices 1, 2, and 3 for *i* or *j* correspond to the Cartesian coordinates (x, y, z) and velocity components (u, v, w).



FIG. 7. (a) As in Fig. 5, but for the 60–120-min average and (b) the tke averaged over the same period; note that the horizontal section is shown at z = 0.2 km. The square in bottom panel of (a) shows the analysis region for Fig. 8.

conclude the  $\overline{\zeta}$  exiting on the downstream side of the square is almost entirely due to the cross-stream flux at y = 0 approximated by  $-\overline{w}\overline{\eta}$ .

From Part I (section 2c) the equation for  $\overline{\eta}$  in statistically steady flow can by be written as

$$\partial_x \overline{Z}_{yx} + \partial_z \overline{Z}_{yz} = -\partial_x \overline{b} - (\partial_x Z'_{yx} + \partial_z Z'_{yz}), \qquad (8)$$

where  $b = g\theta/\theta_{00}$ ,  $(Z_{yx}, Z_{yz}) = (\overline{u}\,\overline{\eta} - \overline{v}\,\overline{\xi} + \nu\nabla^2\overline{w}, \overline{w}\,\overline{\eta} - \overline{v}\,\overline{\zeta} - \nu\nabla^2\overline{u})$  and viscous effects are small as shown below.<sup>3</sup> Consistent with (6), the eddy-correlation term is equivalent to  $\partial_z F_x - \partial_x F_z$ . Figure 10 shows the terms in (8) along y = 0 and the flow vectors; the only place where there is significant baroclinic generation is within the leeside descent region. In the region upstream of the location where  $\overline{\eta}$  reaches a minimum (marked by  $\eta_m$ ) in Fig. 10, the flow is essentially laminar, hence (8) may be written along y = 0 as

$$u\partial_{\mathbf{x}}\boldsymbol{\eta} + w\partial_{\mathbf{z}}\boldsymbol{\eta} = -\partial_{\mathbf{x}}b. \tag{9}$$

Equation (9) can be expressed as  $u_s d\eta/ds = -\partial_x b$  where *s* is the distance and  $u_s$  is the speed along a streamline; this equation may be formally integrated to give  $\eta = \int_s -\partial_x b ds/u_s$ , along a streamline from an upstream point where  $\eta = 0$ . Choosing the streamline that goes down through the zone where  $-\partial_x b < -0.5 \times 10^{-3} \text{ s}^{-2}$  in Fig. 10, one can estimate that  $ds \simeq 500 \text{ m}$  and  $u_s \simeq 5 \text{ m s}^{-1}$  and therefore  $\eta \simeq -\partial_x b ds/u_s \simeq -0.05 \text{ s}^{-1}$ , which is close to the values shown in Fig. 7a.

As one can see in the top panel of Fig. 7a, the field of  $\overline{\eta}$  becomes more diffuse with distance downstream; thus  $\overline{w} \overline{\eta}$  is nearly zero at z = 0.4 km and therefore cannot cancel out the positive contributions from the downward transport of  $\overline{\eta} < 0$  in the integral on the rhs of (5). In the turbulent layer below  $z \simeq 0.4$  km, Fig. 10 shows that the main balance in (8) is between mean and turbulence transport of  $\overline{\eta}$ . For clarity, Fig. 11 shows the *x*-averaged values of the terms in (8) which indicates that the time-mean downstream transport of negative  $\overline{\eta}$  (<0) is balanced by turbulent diffusion (>0). A similar analysis of (6) downstream of the

<sup>&</sup>lt;sup>3</sup> Note that  $Z_{yz} = -Z_{zy}$  as a consequence of the antisymmetry of the vorticity flux tensor  $Z_{ij}$  (Haynes and McIntyre 1987).



FIG. 8. Vertical vorticity  $\overline{\zeta}$  (shading) and vertical vorticity flux vectors ( $Z_{zx}$ ,  $Z_{zy}$ ) at z = 0.4 km in the analysis square shown in Fig. 7a based on (a) the exact form (2) and (b) the approximate form ( $\overline{u}\overline{\zeta}$ ,  $-\overline{w}\overline{\eta}$ ).

square shown in Fig. 7a (not shown) indicates a balance of time-mean downstream advection and turbulent diffusion of  $\overline{\zeta}$ .

In summary, the analysis leading to (7) shows that the vertical vorticity  $\overline{\zeta}$  of the lee vortices is directly related to the time-mean downward transport of negative spanwise vorticity  $\overline{w} \overline{\eta} > 0$  while the analysis of the flow using (9) and Fig. 10 show that the downwardtransported negative  $\overline{\eta}$  is baroclinically generated. Therefore the vertical vorticity of the lee vortices  $\overline{\zeta}$  must originate through the inviscid, adiabatic process of baroclinic generation of spanwise vorticity  $\overline{\eta}$  in the leeside descending stratified flow. Turbulent diffusion of  $\overline{\eta}$ downstream is important as it reduces contributions from negative values of  $\overline{w} \overline{\eta}$  that would, in the absence of turbulent diffusion, bring the integral on the rhs of (5) to zero, and therefore, the leeside vertical vorticity  $\zeta$  would not extend downstream. This conclusion is consistent with that from Rotunno et al. (1999, section 5) based



FIG. 9. (a) Cross-stream flux of vertical vorticity  $Z_{zy}|_{y=0}$  and (b) averaged downstream flux of vertical vorticity  $\langle Z_{zx} \rangle$  using the exact form (black lines) and the approximate form (green lines) corresponding to the integrands in (7).

on a simulation of lee vortices in a continuously stratified fluid with parameterized turbulent diffusion.

Before leaving this section, the conclusion from the analysis of (6) that the Reynolds stress terms are small allows a simple computation of the terms in the advective form of the vertical vorticity equation in terms of the time-mean fields, namely,

$$\overline{\mathbf{u}} \cdot \nabla \zeta = \overline{\xi} \partial_x \overline{w} + \overline{\eta} \partial_y \overline{w} + \overline{\zeta} \partial_z \overline{w}$$
(10)

where the first two terms on the rhs are called the "tilting" terms and the last term on the rhs is called the "stretching" term. Figure 12 shows the tilting and stretching terms at z = 0.2 and z = 0.4 km for the statistically steady flow; the stretching term is dominant at z = 0.4 km while the tilting term is negative; at z = 0.2 km, the tilting term is positive and dominant while the stretching term is small. These outcomes are expected based on the analysis of Epifanio and Durran (2002) of a leeside breaking mountain wave (identified as a hydraulic jump) in a continuously stratified flow. Their analysis of fluid-parcel trajectories passing



FIG. 10. Budget for  $\overline{\eta}$  at y = 0 with the time-mean transport term,  $-\partial_x \overline{Z}_{yx} - \partial_z \overline{Z}_{yz}$ , in blue-to-red color shading; the eddy-flux term,  $-\partial_x Z'_{yx} - \partial_z Z'_{yz}$ , in the color contours (-0.002, -0.001, and -0.005 in blue and 0.0005, 0.001, and 0.002 in red); and the baroclinic term,  $-\partial_x \overline{D}$ , indicated by the single dashed black contour line of  $-0.5 \times 10^{-3} \, \text{s}^{-2}$ . The  $\eta_m$  on the leeside of the obstacle indicates the location of the negative minimum of  $\overline{\eta}$ . The red line corresponds to the location of the square shown in the bottom panel of Fig. 7a. The rectangle indicates the window for the *x*-averaged terms shown in Fig. 11.

through the leeside vertical vorticity maxima indicate that during the parcel's leeside descent, it acquires horizontal vorticity through baroclinicity which is subsequently tilted to become vertical vorticity which is, in turn, stretched at the across-stream end points of the hydraulic jump (Epifanio and Durran 2002, p. 1174). The next section will provide further information on the structure of the vorticity distribution in the present simulated hydraulic jump.

## b. Potential vorticity

With the present direct numerical solutions of the NSE, we are in position to make a connection between the foregoing analysis of the lee-vortex  $\boldsymbol{\omega}$  and PV. Following Epifanio and Qian (2008, section 5b), we look at the potential vorticity (1) defined as

$$\overline{\mathbf{PV}} = \overline{\boldsymbol{\omega}} \cdot \nabla \overline{\boldsymbol{\theta}}. \tag{11}$$

We note that the potential vorticity as defined in (11) is the quantity whose conservation equation results from the curl of the time-averaged momentum equation and the gradient of the time-averaged potential-temperature equation. Figure 13 shows  $\overline{PV}$  at z = 0.4 km in the same window as the bottom panel of Fig. 7a. As expected from the SWE solutions (Fig. 1b), there is a dipole of  $\overline{PV}$  associated with the lee vortices.



FIG. 11. Budget for  $\overline{\eta}$  at y = 0 averaged from x = -0.5 to x = 0.5 km as a function of z showing the transport due to the timemean fields,  $-\partial_x(\overline{u} \ \overline{\eta} - \overline{v} \ \overline{\xi}) - \partial_z(\overline{w} \ \overline{\eta} - \overline{v} \ \overline{\xi})$ ; the time-mean eddy term,  $-\partial_x(\overline{u' \eta'} - \overline{v' \xi'}) - \partial_z(\overline{w' \eta'} - \overline{v' \xi'})$ ; the baroclinic term,  $-\partial_x \overline{b}$ ; and the viscous term,  $\nu(\partial_{xx} + \partial_{zz})\overline{\eta}$ , where  $\nu = 1 \text{ m}^2 \text{ s}^{-1}$ .

The evolution equation for  $\overline{PV}$  is

$$\partial_{\cdot} \overline{\mathbf{PV}} + \nabla \cdot \mathbf{J} = 0 \tag{12}$$

where the  $\overline{PV}$ -flux vector is defined as

$$\mathbf{J} = \overline{\mathbf{u}} \,\overline{\mathbf{PV}} + \nabla \overline{\boldsymbol{\theta}} \times \mathbf{F} - \overline{\boldsymbol{\omega}} \,\overline{Q} \,. \tag{13}$$

[Epifanio and Qian 2008, their Eq. (24)]; in this definition **F** is (minus) the divergence of eddy-momentum flux and Q is (minus) the divergence of the eddy-heat flux. As with the equation for  $\overline{\zeta}$ , the y component of **J** is a central element in understanding the origin of the  $\overline{PV}$ dipole. For steady-state flow, (12) reduces to  $\nabla \cdot \mathbf{J} = 0$ and therefore a cross stream flux  $J_y < 0$  at y = 0 is required to maintain a  $\overline{PV}$  dipole downstream with  $\overline{PV} > 0$ for y < 0. At y = 0, we expect by symmetry that  $\overline{v} = 0$  and the equation for  $\overline{\theta}$  is

$$\overline{u}\partial_{x}\overline{\theta} + \overline{w}\partial_{z}\overline{\theta} = \overline{Q}.$$
(14)

Under these conditions we have at y = 0

$$J_{y} = (F_{x} - \overline{w}\,\overline{\eta})\partial_{z}\overline{\theta} - (F_{z} + \overline{u}\,\overline{\eta})\partial_{x}\overline{\theta}.$$
(15)

Figure 14a shows that  $J_y(y = 0)$  reaches a negative minimum along the axis of the maximum downward excursion of the leeside isentropes. Figures 14b and 14c show the frictional and heating contributions to  $J_y, J_y^F$ , and  $J_y^Q$ . Although these two components both contribute to the magnitude of  $J_y$ , the structure  $J_y^Q$ seems to best explain that of  $J_y$ . Figure 14 taken together with (15) suggests a close connection with the analysis of the  $\overline{\zeta}$  from Fig. 9, which is the importance of time-mean transport of  $\overline{\eta}$ . In the following we explore this connection from a geometrical point of view.



FIG. 12. Tilting and stretching terms in the advective form of the vertical vorticity Eq. (10) at (a),(b) z = 0.4 km and (c),(d) z = 0.2 km. Contours are plotted in increments of  $1 \times 10^{-4}$  s<sup>-2</sup> (zero line not shown, negative values denoted by dashed lines) while the color shading shows the time-averaged vertical vorticity. The thick blue line in (c) and (d) shows the outline of the obstacle at z = 0.2 km.

The definition PV in (1) is a measure of the degree to which  $\boldsymbol{\omega}$  intersects surfaces of constant  $\theta$ , or in other words, the degree to which  $\theta$  varies along a vortex line, namely,

$$PV = |\boldsymbol{\omega}| \frac{\delta\theta}{\delta l}, \qquad (16)$$

where  $\delta l$  is an infinitesimal distance along a vortex line. The following three-dimensional analysis makes extensive use of (16), but using  $\overline{\omega}$  and  $\overline{\theta}$ .

Figure 15 shows a series of vortex lines that go through the locations of minimum  $\overline{\eta}$  at y = 0 (see the top panel of Fig. 7a); the color of the vortex line indicates the local value of  $\overline{\theta}$ . Vertical (x-z) cross sections of  $\overline{\theta}$  (shaded) for z < 1.25 km in the planes at y = 0 and y = 1.9 km allow one to see the location and  $\overline{\theta}$  of three of the vortex lines as they enter the display domain at y = 1.9 km and  $z \approx$ 1 km (the height of the undisturbed layer of strong potential-temperature increase shown in Fig. 2) and then descend for y > 0 (implying  $\overline{\zeta} < 0$ ) and then ascend for y < 0 (implying  $\overline{\zeta} > 0$ ). All four of these vortex lines display little variation of  $\overline{\theta}$  along their lengths and thus, according to (16),  $\overline{PV} \approx 0$ .

In contrast, Fig. 16 shows a vortex line that passes through the  $\overline{PV}$  extrema of Fig. 13. In this case,  $\overline{\theta}$  increases along the vortex line ( $\overline{PV} > 0$ ) for y < 0 and decreases along the vortex line ( $\overline{PV} < 0$ ) for y > 0. Note that, as with the vortex lines of Fig. 15, this vortex line passes close to location of minimum  $\overline{\eta}$  at y = 0, which indicates the basic baroclinic source of  $\overline{\eta}$  is common to



FIG. 13.  $\overline{\text{PV}}$  as defined in (11) at z = 0.4 km along with the vectors  $(\overline{u}, \overline{v})$  and the isoline of  $\overline{\theta} = 289$  K (green contour).

vortex lines with and without significant  $\overline{PV}$ . Note also that the vortex line in this case forms a loop that connects the lower distribution of  $\overline{\eta} < 0$  to the upper distribution where  $\overline{\eta} > 0$  shown in the upper panel of Fig. 7a. Also shown for comparison is a vortex line from Fig. 15.

Figure 17 shows the vortex lines from Fig. 16 together with the isosurface  $\overline{\theta} = 289$  K. Figure 17a suggests the following interpretation. The baroclinically produced  $\overline{\eta} < 0$ , which forms the lower portion of these vortex lines, induces the spanwise rotary motion that continually transports the isosurface of  $\overline{\theta}$  back toward the obstacle, thus wrapping the isosurface around the lower portion of the vortex line and thereby producing a  $\overline{\theta}$ gradient along it. The vortex line thus descends through the  $\overline{\theta}$  surface for y > 0 and ascends through it for y < 0(Fig. 17b) which, according to (16), implies a dipole in  $\overline{PV}$ .

The vortex lines shown in Fig. 15 are consistent with those found in the inviscid, adiabatic linear theory of Smolarkiewicz and Rotunno (1989, their Fig. 2b); however, the one in Fig. 16 is not. As demonstrated above, this vortex line loops more or less vertically through an isentropic surface and thus has significant gradients of potential temperature along its length and therefore significant potential vorticity. To understand the origin of this looping vortex line, we examine the vortex lines and isosurfaces of  $\theta$  at the time when the leeside reversed flow first forms (Fig. 3b).

Figure 18 shows several vortex lines passing through y = 0 at t = 8 min together with  $\theta(x, 0, z)$  and  $\theta(x, y = 1.9 \text{ km}, z)$ . Each vortex line has little variation of  $\theta$  along its length and so by (16), PV = 0. Figure 19 shows the same vortex lines together with the isosurface  $\theta = 289$  K.



FIG. 14. Cross-stream flux of  $\overline{PV}$  at y = 0 for (a)  $J_y$ , (b)  $J_y^F$  (the frictional part of  $J_y$ ), and (c)  $J_y^Q$  (the heat-transfer part of  $J_y$ ). Vectors and isentropes are as in the top panel of Fig. 7a.

These figures together suggest that the looping, potentialvorticity-bearing vortex line seen in the steady-state configuration (Figs. 16, 17) has its origin in the inviscid, adiabatic, but nonlinear, fluid dynamics of wave overturning, which in turn, lead to local static instability and turbulence. PV generation occurs subsequently as the crest of the wave moves through the loop and reconnects through turbulent mixing with fluid in the wave trough. As discussed in relation to Figs. 3c and 3d, this process of wave formation and overturning occurs repeatedly through the simulation and thus the vortex-loop structure



FIG. 15. Vortex lines that go through the strip of minimum  $\overline{\eta}$  at y = 0 shown in the top panel of Fig. 7a. Vertical (x-z) cross sections of  $\overline{\theta}$  (shaded) for z < 1.25 km in the planes y = 0 and y = 1.9 km are shown with the color scale in the upper left. Color of the vortex line varies along their lengths according to the same color scale. These vortex lines have only small variations of  $\overline{\theta}$  which, according to (16), indicates small values of  $\overline{PV}$  along its length. The horizontal display domain is as in Fig. 13 ( $-3 \le x \le +1$  km,  $-2 \le y \le +2$  km).

emerges in the time average of the statistically stationary solution.

## c. The case where an obstacle pierces the two-layerfluid interface in the SWE

Epifanio and Rotunno (2005) consider the case where the fluid is stably stratified below the height of the obstacle crest with much weaker stratification above; this case is the continuous-fluid analog to the pierced-layer case in the SWE (Figs. 1c,d). Figure 20 (Epifanio 2014) depicts the evolution of lee vortices in this flow started from rest; the flow regime is one in which the upstream flow below the obstacle crest is blocked and must flow around, rather than over, the obstacle. Figure 20 shows the vortex lines on an evolving isentropic surface with centers of leeside vertical vorticity forming as the isentropes and vortex lines bend downward on the leeside flanks of the obstacle. We note that in this case the vortex lines associated with the return flow loop over it and begin and terminate at the terrain. These experiments show that lee vortices can also form for cases in which the obstacle extends above the stratified layer with PV =0. PV is subsequently created by diffusive effects as the evolving isentropes of each vortex wrap in a spiraling motion. While the evolution of this flow differs from that of the present case of flow of the stratified layer over the obstacle, the conclusion is the same: lee vortices can form under conditions of PV =0; PV is created subsequently as a result of the motions engendered by the vorticity distributions created by the inviscid, adiabatic processes in a density-stratified fluid.

We note that the "looping-over" structure of the vortex lines has been commented on in Schär and



FIG. 16. As in Fig. 15, but for a vortex line that has a large variation of  $\overline{\theta}$  along its length indicating large  $\overline{PV}$  where the variations are largest according to (16). Also included for comparison is a vortex line from Fig. 15.

Durran (1997, their Fig. 12) and in Rotunno et al. (1999, their Fig. 4) in their respective analyses of continuously stratified flow. The structure of these solutions in the y-z plane suggests that the lee vortices in these cases are associated with blocking of the upstream flow [Fig. 4c in Schär and Durran (1997) and Fig. 3a in Rotunno et al. (1999)].



FIG. 17. Vortex lines from Fig. 16 together with the isosurface  $\overline{\theta} = 289 \,\text{K}$  from a downstream viewpoint looking (a) down at an angle to the flow and (b) from a lower level, more directly into the flow.



FIG. 18. Vortex lines that go through the incipient wave-breaking zone at y = 0 at  $t = 8 \min$  (Fig. 3b). Vertical (x-z) cross sections of  $\theta$ (shaded) for z < 1.25 km in the planes y = 0 and y = 1.9 km are shown with the color scale in the lower left. Color of the vortex line varies along their lengths according to the same color scale. These vortex lines have only small variations of  $\theta$ , which, according to (16), indicates small values of PV along its length. The horizontal display domain is as in Fig. 15.

## 4. Discussion and conclusions

As a model for atmospheric motions, the shallowwater equations (SWE) have two limitations that complicate interpretation in terms of concepts based on the full Navier–Stokes equations (NSE). First, the SWE treat fluids with layers of constant density which implies a discontinuity in the vertical derivative of density at the interface between the layers. Second, these equations are only valid for motions with horizontal length scales much greater than vertical length scales.

The steepening of the height of the fluid layer h(x, t)toward a discontinuity (interpreted as the hydraulic jump; see Fig. 2 of Part I) in the SWE violates the second limitation. The formation of a discontinuity indicates a breakdown of some approximation to the physics; to "save" the solution one typically allows the discontinuity and then matches the solutions on either side based on physical principles to obtain a "weak" solution (Whitham 1974, p. 26). In the case of a hydraulic jump, the conservation of mass and momentum in the layer provide the matching conditions (Whitham 1974, 458-460). Alternatively, Schär and Smith (1993, their section 2e) formulate a diffusion term that prevents a discontinuity while conserving mass and momentum. With either method, energy is dissipated through the zone of large height and velocity gradients, which leads to the creation of shallow-water-equation



FIG. 19. Vortex lines from Fig. 18 together with the isosurface  $\theta = 289$  K from a viewpoint looking down and across flow.

potential vorticity,  $\zeta/h$ , and therefore, vorticity  $\zeta$ , since h > 0. Hence their conclusion that dissipation creates leevortex vorticity (Schär and Smith 1993, their section 2d).

As with the two-dimensional case of Part I, the present study of flow past a three-dimensional obstacle produces a leeside spanwise rotary motion in the alongstream-vertical plane but of limited spanwise extent and therefore, we have shown, lee vortices. Ertel potential vorticity (PV) is produced as a consequence of the dissipative processes that result from the frictionless, adiabatic dynamics giving rise to leeside rotary motion/lee vortices. Hence the view from the NSE is that the threedimensional flow associated with lee vortices leads to PV production, not vice versa.

The reconciliation of these contrasting conclusions from the SWE and the NSE concerning lee vortices and potential vorticity production can be summarized as follows. The internal dynamics of the hydraulic jump is not accessible to the SWE since irreversible processes must be parameterized at the leeside discontinuity (identified with the physical hydraulic jump) (Schär and Smith 1993, their section 2e). The potential vorticity in the SWE, and therefore the vorticity, is created by the parameterized dissipation at the discontinuity. Hence, the view of lee-vortex formation from the SWE focuses on the leeside vorticity downstream of the discontinuity/ hydraulic jump (Fig. 1b). Solutions to the NSE, on the other hand, reveal the hydraulic jump's internal dynamics is driven by frictionless, adiabatic processes that result in the overturning of the thin transition layer which leads to local static instability and turbulence (Fig. 11 of Part I). With the present simulation using a three-dimensional obstacle, the rotary motion/reversed flow has a limited spanwise extent, which we have shown is consistent with a leeside dipole in the vertical vorticity. The latter turbulent motions give rise to energy dissipation which, consistent with the SWE, generate the potential vorticity that flows downstream from the hydraulic jump.





FIG. 20. Evolution of an isentropic surface with vortex lines lying in the surface for a flow with the stratified layer below the obstacle top. Colors on the surface indicate the perturbation temperatures (blues: negative; oranges: positive). In this case the vortex lines of the evolving lee vortices loop over the reversed flow and terminate at the lower surface. Figure 6 in Epifanio (2014) modified from Epifanio and Rotunno (2005). Arrows added to indicate the sense of the vortex lines.

In summary, the principal conclusions are as follows:

- Analysis of the NSE indicates the original source of the vertical vorticity of the lee vortices is the descent of negative spanwise vorticity, expressed analytically in (7). The role of turbulent dissipation, emphasized in the SWE, prevents the re-ascent of negative spanwise vorticity that would limit the downwind transport of vertical vorticity.
- Analysis of the NSE shows that leeside negative spanwise vorticity is created baroclinically in the descending leeside flow [Fig. 10 and (9)]. Figure 15 shows that tilting of the vortex lines with negative spanwise vorticity produces the dipole of vertical vorticity associated with the lee vortices. In the SWE, and more generally, in typical formulations of water-wave equations (Whitham 1974, chapter 10), baroclinic vorticity generation does not appear explicitly, but rather, is implicit at the two-layer interface (Rotunno and Smolarkiewicz 1995, their section 2c).

- The leeside hydraulic jump is critical to lee-vortex formation in SWE as it implies energy dissipation and the creation of a potential-vorticity dipole. The internal dynamics is not, however, accessible to the SWE. According to the NSE, the hydraulic jump is a backward overturning isentrope with the reversed flow induced by baroclinically generated negative spanwise vorticity. The static instability and mixing induced by the overturning isentrope leads to energy dissipation and potential-vorticity creation.
- Analysis of the NSE leads to a geometrical interpretation of the potential-vorticity dipole associated with the lee vortices in which the overturning isentropes envelope the baroclinically produced spanwise vorticity (Fig. 17); this process originates in the inviscid, adiabatic, nonlinear dynamics leading to isentrope overturning (Fig. 19).

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