Dynamic Mesoscale Mountain Meteorology

Lecture 3: Mountain Waves

Richard Rotunno

National Center for Atmospheric Research , USA

Topics

Lecture 1 : Introduction, Concepts, Equations

Lecture 2: Thermally Driven Circulations

Lecture 3: Mountain Waves

Lecture 4: Mountain Lee Vortices

Lecture 5: Orographic Precipitation

Mountain Waves

Mt. Shasta

Jane English

Wave Clouds at Three Levels

Bob Houze

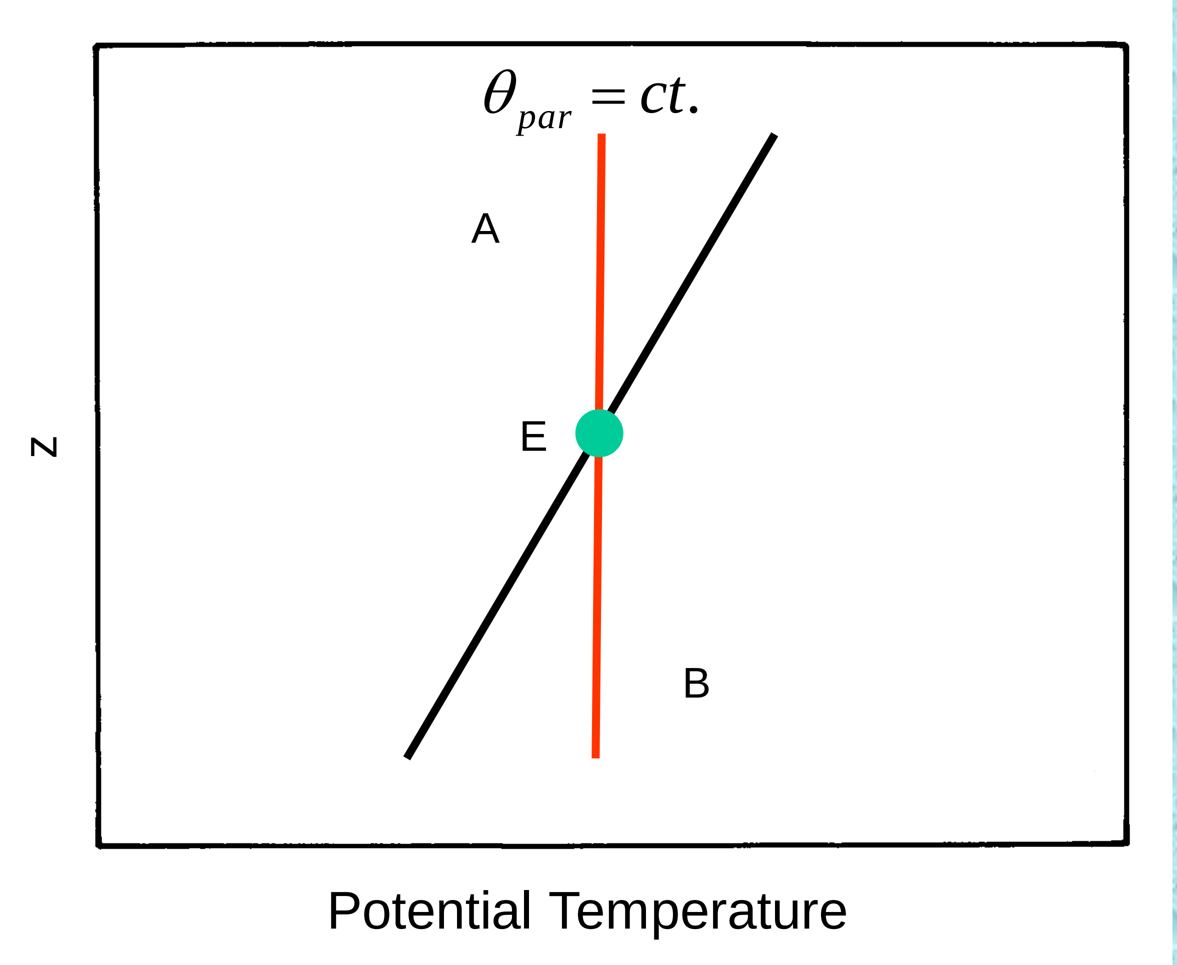
Multiple Wavelength Wave Clouds

Dane Gerneke

Rotor Cloud in lee of the Sierra Nevada

R. Symons

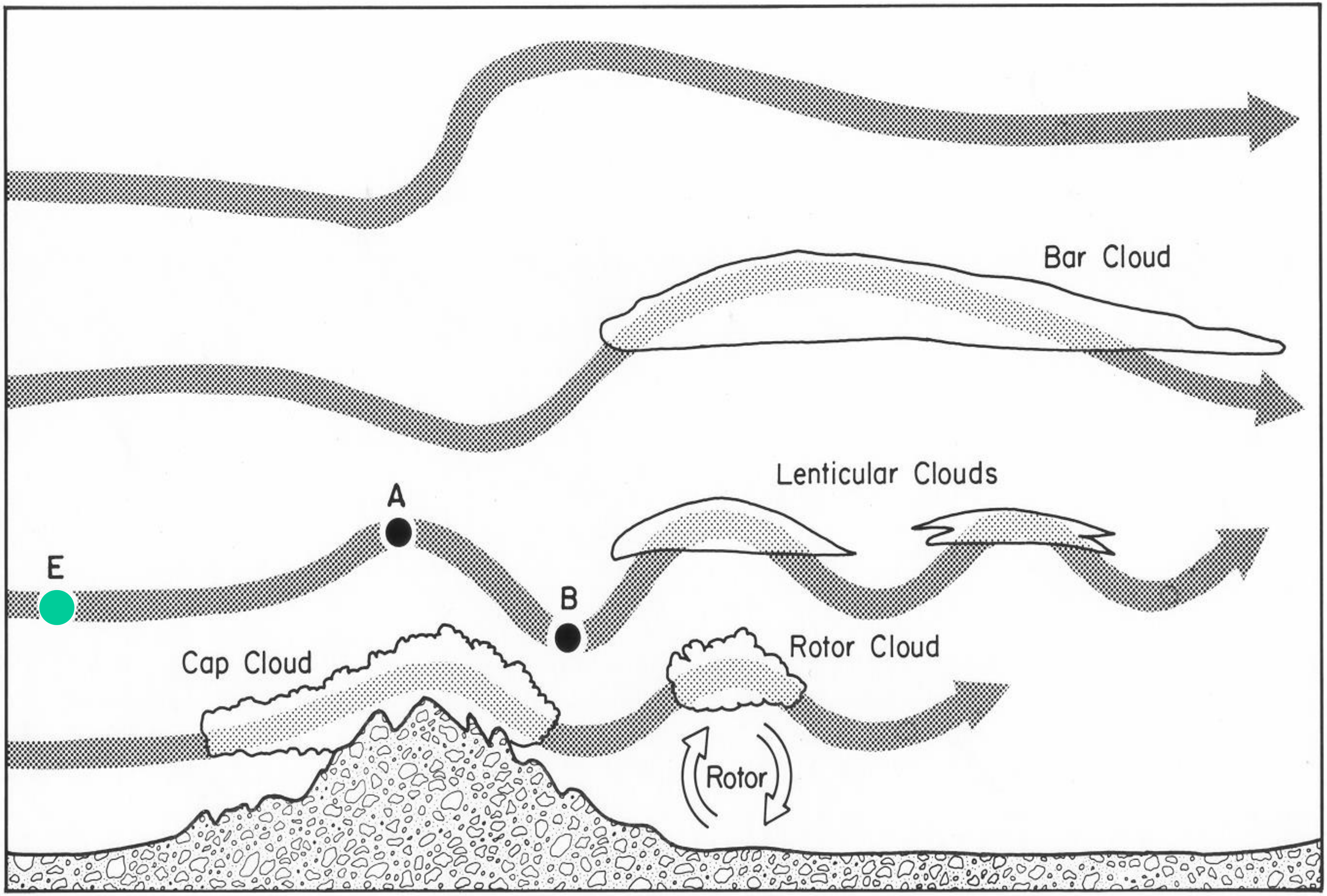
Air Parcel Behavior in a Stable Atmosphere



$$\ddot{\delta} = B = -N^2\delta \Rightarrow \delta \sim e^{\pm iNt}$$

$$N^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z}\bigg|_{env} > 0$$

Mountain Waves under Stable Conditions



Back to Inviscid, Adiabatic Governing Equations

Conservation of

momentum

$$\frac{D\vec{u}}{Dt} = -\nabla\varphi + B\hat{k} + \nu\nabla^2\vec{u}$$

energy

$$\frac{DB}{Dt} = \kappa\nabla^2 B$$

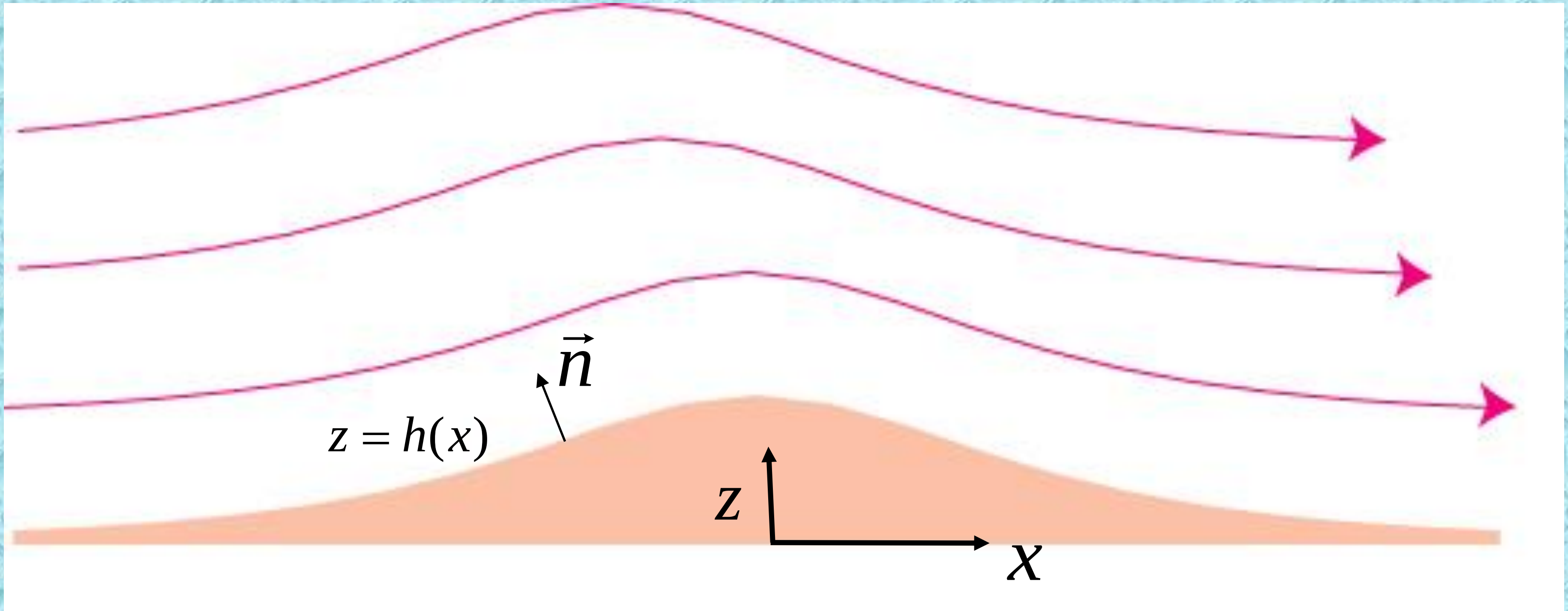
mass

$$\nabla\cdot\vec{u} = 0$$

Flow Past an Impermeable 2D Obstacle

$$\vec{u}\cdot\vec{n}=0 \quad \text{on } z=h(x) \Rightarrow$$

$$w=\left(u\frac{\partial h}{\partial x}\right)_{z=h(x)}$$



Vorticity Equation

Lecture 2 →

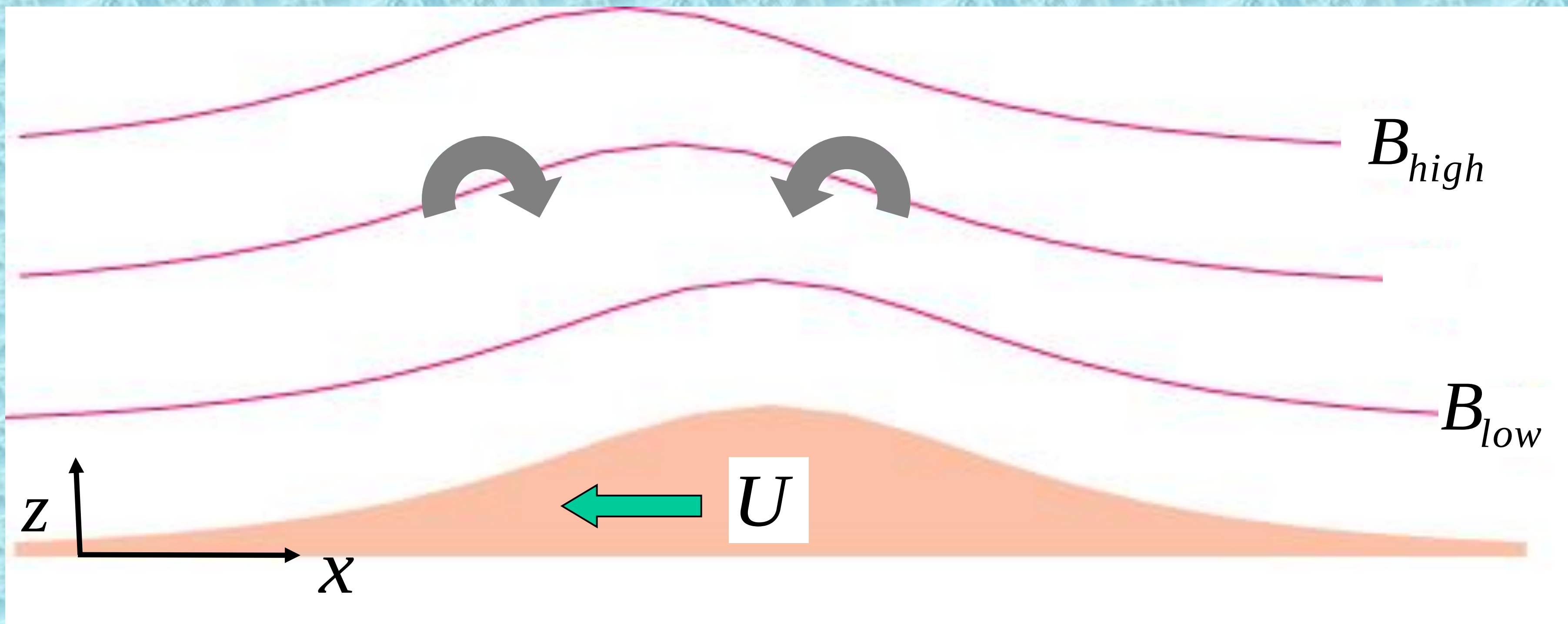
$$\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\vec{u} - \hat{k}\times\nabla B$$

$$\vec{\omega} = (\xi, \eta, \zeta)$$

2D version →

$$\frac{D\eta}{Dt} = -\partial_x B$$

Baroclinicity



2D Eqns in Terms of Vorticity and Streamfunction

Lecture 2 →

$$u = \partial_z \psi \;,\; w = -\partial_x \psi \quad ; \quad \eta = \nabla^2 \psi$$

vorticity

$$\frac{D}{Dt} \nabla^2 \psi = -\partial_x B$$

energy

$$\frac{DB}{Dt} = 0$$

$$\frac{D}{Dt} = \partial_t + \partial_z \psi \, \partial_x - \partial_x \psi \, \partial_z$$

Simplify

$$B = N^2 z + b$$

vorticity

$$\frac{D}{Dt}\nabla^2\psi = -\partial_x b$$

energy

$$\frac{Db}{Dt} = N^2\partial_x\psi$$

$$\frac{D}{Dt} = \partial_t + \partial_z\psi\,\partial_x - \partial_x\psi\,\partial_z$$

2D Linear Theory of Internal Gravity Waves

vorticity

$$\partial_t \nabla^2 \psi = -\partial_x b$$

energy

$$\partial_t b = N^2 \partial_x \psi$$

combine vorticity
and energy eqns→

$$\partial_t^2 \nabla^2 \psi + N^2 \partial_x^2 \psi = 0$$

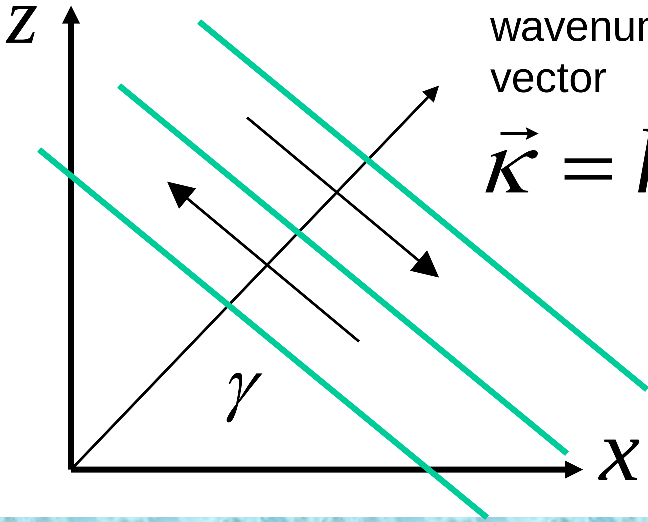
Lighthill (1978, Chap. 4)

plane-wave solution

$$\psi \propto \psi_c \exp\left[i\left(kx + mz - \omega t\right)\right]$$

$$\omega = \pm \frac{Nk}{\sqrt{k^2 + m^2}} = N\left|\cos\gamma\right|$$

$$0 \le \omega \le N$$



wavenumber vector

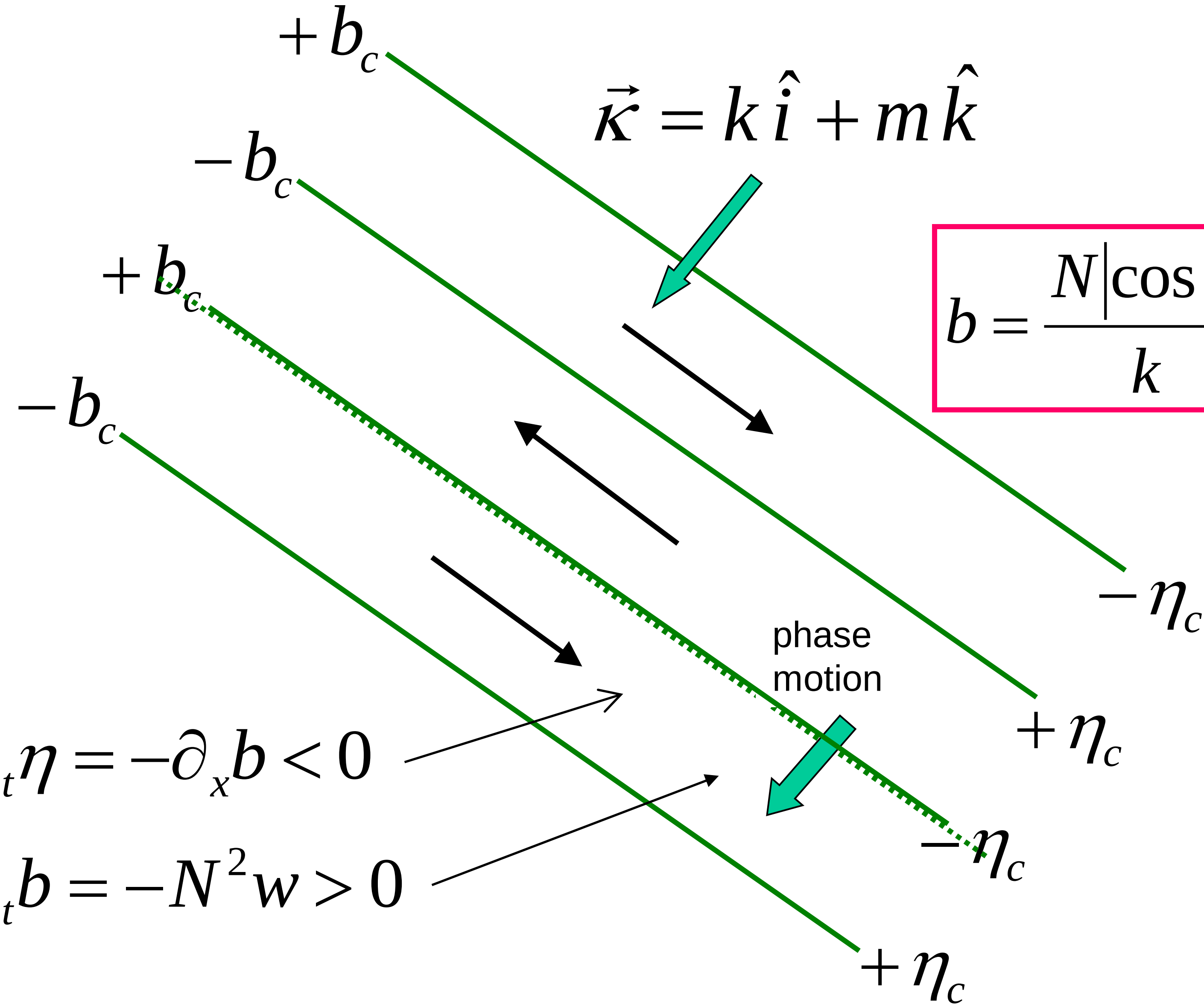
$$\vec{\kappa} = k\,\hat{i} + m\,\hat{k}$$

phase velocity

$$\vec{c} \parallel \vec{\kappa}$$

$$\vec{c} = \frac{\omega}{\kappa}\left(\frac{k}{\kappa}, \frac{m}{\kappa}\right) = \frac{N\left|\cos\gamma\right|}{\kappa^2}(k, m)$$

Example



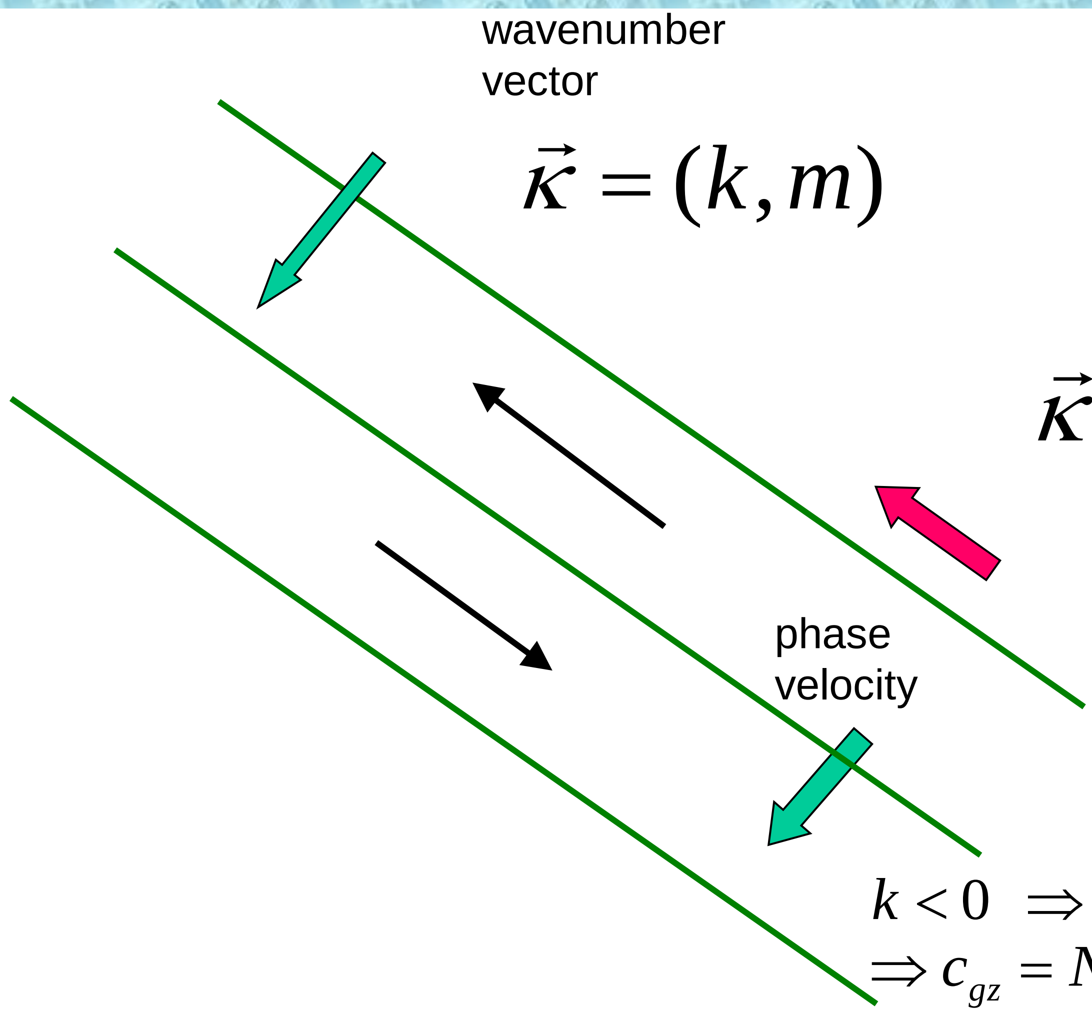
$$\partial_t \eta = -\partial_x b < 0$$

$$\partial_t b = -N^2 w > 0$$

Internal Gravity Waves: Group Velocity

$$\vec{c}_g = \left(\partial_k \omega, \partial_m \omega\right) = \pm \frac{Nm}{\left(k^2 + m^2\right)^{3/2}} (m, -k)$$

Lighthill (1978, Chaps. 3-4)



$$\vec{\kappa} = (k, m)$$

$$\vec{\kappa} \cdot \vec{c}_g = 0$$

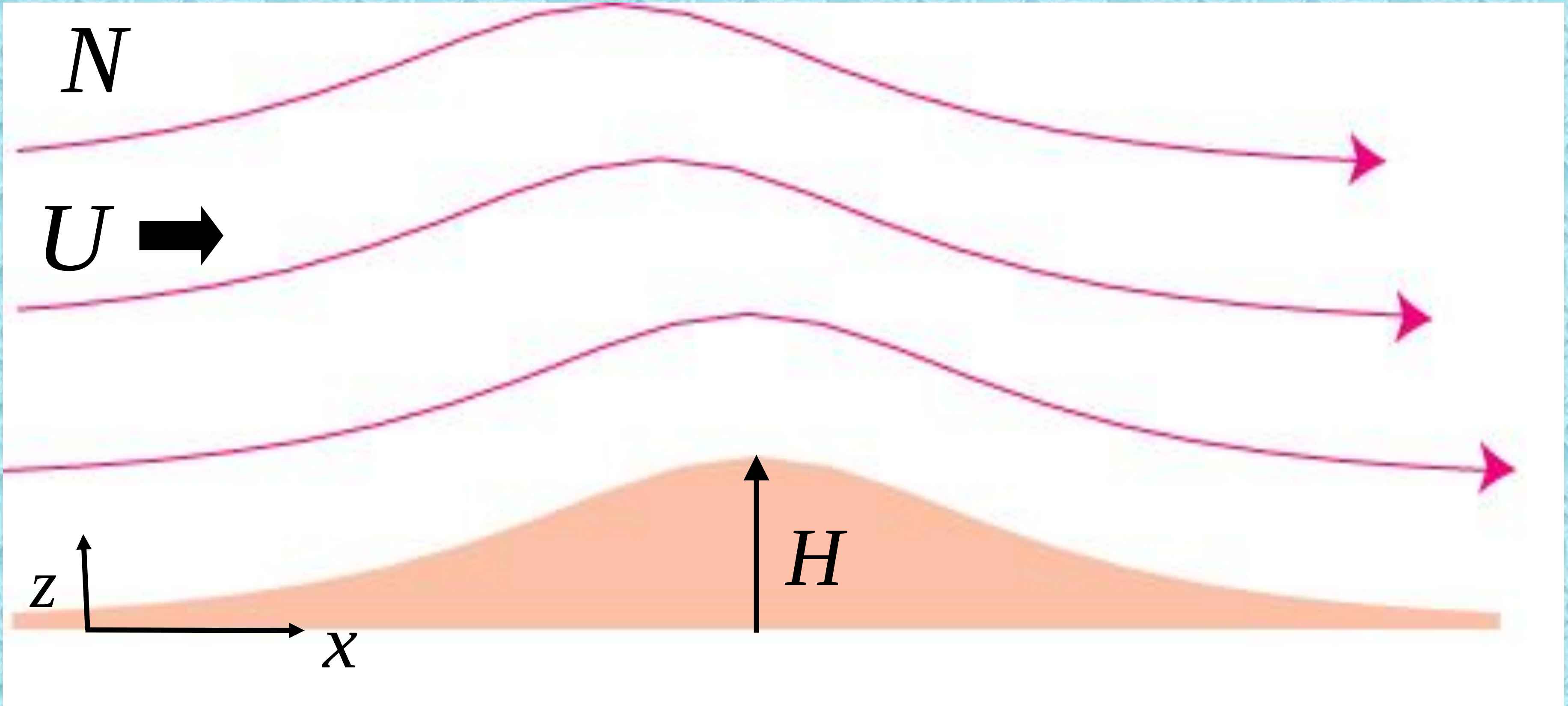
In this example

$$k < 0 \;\Rightarrow \omega = -Nk(k^2 + m^2)^{-1/2}$$

$$\Rightarrow c_{gz} = Nmk(k^2 + m^2)^{-3/2} > 0 \text{ with } m < 0$$

Linear Theory of Mountain Waves

$$H << \frac{U}{N} \sim \frac{10m/s}{.01/s} = 1000m$$



Linear Theory of Mountain Waves

$$\partial_t \to U\partial_x$$

vorticity

$$\partial_t \nabla^2 \psi = -\partial_x b$$

$$U\nabla^2 \psi = -b$$

energy

$$\partial_t b = N^2 \partial_x \psi$$

$$Ub = N^2 \psi$$

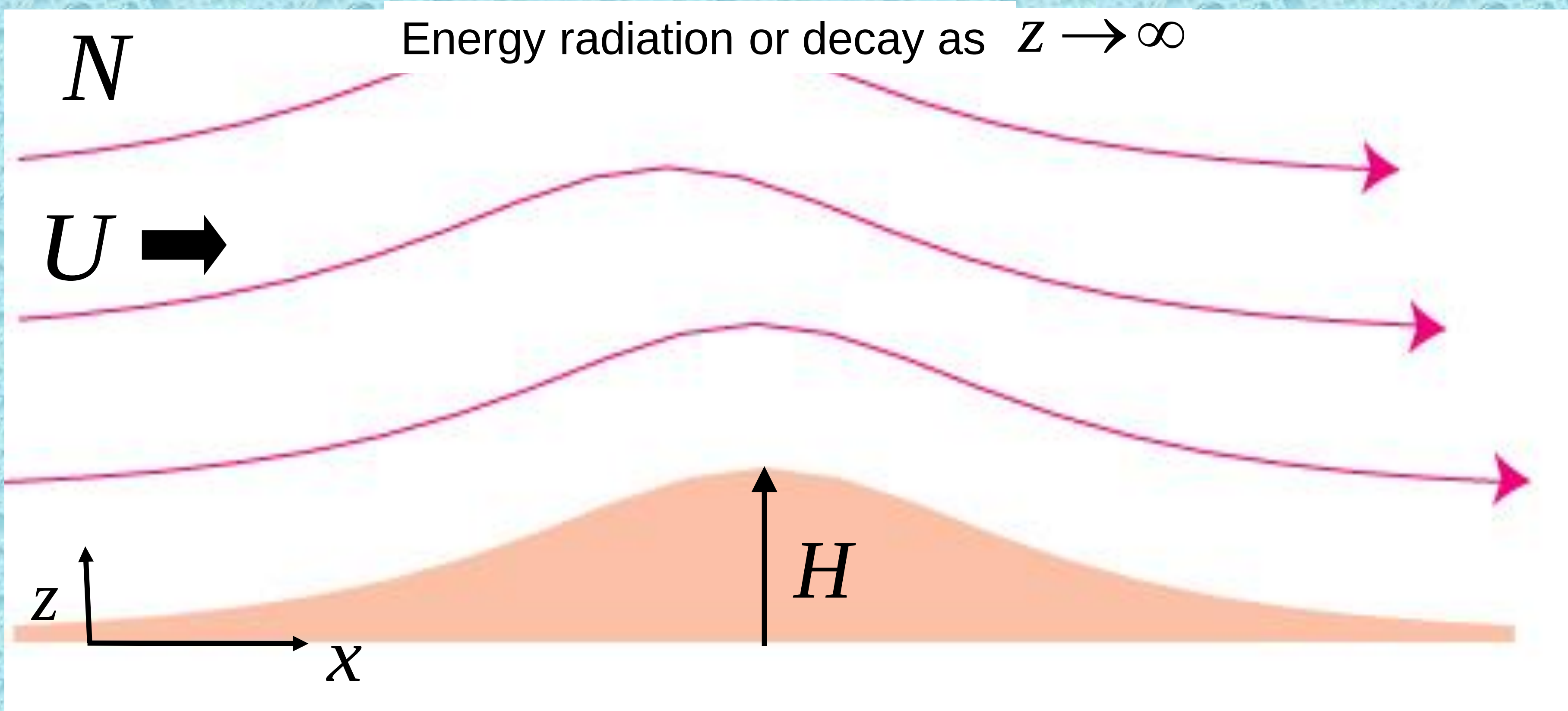
combine→

$$\partial_t^2 \nabla^2 \psi + N^2 \partial_x^2 \psi = 0$$

$$U^2 \nabla^2 \psi + N^2 \psi = 0$$

Equation + Boundary Conditions

$$U^2\nabla^2\psi + N^2\psi = 0$$



$$w = \left(u\frac{\partial h}{\partial x} \right)_{z=h(x)} \Rightarrow \psi = -Uh(x) \text{ at } z = 0$$

Example: Sinusoidal Hill

$$h(x) = H\cos\alpha x \quad ; \quad \alpha \equiv \frac{2\pi}{L}$$

b.c. →

$$\psi(x, z=0) = -\tfrac{1}{2}UH(e^{+i\alpha x} + e^{-i\alpha x})$$

Look for sol'n of the form→

$$\psi(x,z) = \widehat{\psi}_{+}(z)e^{+i\alpha x} + \widehat{\psi}_{-}(z)e^{-i\alpha x}$$

Gov'n eqn →

$$\frac{\partial^2 \widehat{\psi}_{\pm}}{\partial z^2} + \left(\frac{N^2}{U^2} - \alpha^2\right)\widehat{\psi}_{\pm} = 0$$

Case 1 →

$$\frac{N^2}{U^2} - \alpha^2 > 0$$

Sol'n →

$$\widehat{\psi}_+(z) = -\tfrac{1}{2} UHe^{\pm i\beta z} \;,\; \widehat{\psi}_-(z) = -\tfrac{1}{2} UHe^{\pm i\beta z}$$

$$\beta \equiv \left(\frac{N^2}{U^2} - \alpha^2 \right)^{\frac{1}{2}} > 0$$

use radiation condition to indicate which sign to choose in the exponents

dispersion relation including *U*

$$\omega - Uk = \pm \frac{Nk}{\sqrt{k^2 + m^2}}$$

for steady sol'n $\omega = 0$ must choose minus sign in dispersion relation and so

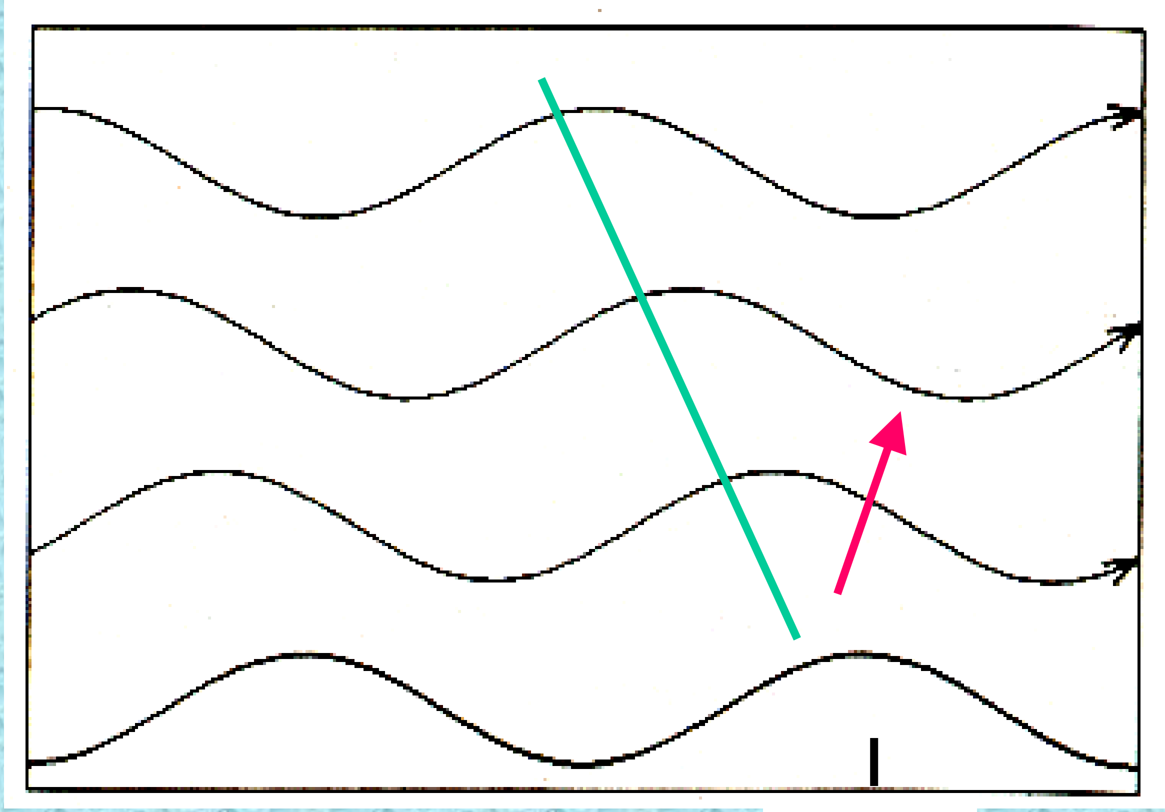
$$c_{gz} = \frac{Nmk}{\left(k^2 + m^2\right)^{3/2}}$$

$$m = \mathrm{sgn}(k) \quad \Rightarrow \quad c_{gz} > 0$$

$$k = +\alpha \rightarrow m = +\beta \ \ ; k = -\alpha \rightarrow m = -\beta$$

$$\hat{\psi}_+(z) = -\tfrac{1}{2} U H e^{+i\beta z} \ , \hat{\psi}_-(z) = -\tfrac{1}{2} U H e^{-i\beta z}$$

$$\psi(x,z) = -UH\sin(\alpha x + \beta z)$$



$$x = 0$$

$$-Uk = -\frac{Nk}{\sqrt{k^2+m^2}}$$

Doppler-shifted=Natural Freq

$$c_{gz} = U\left(\frac{U}{N}\right)^2 mk > 0$$

$$c_{gx} = U\left(\frac{U}{N}\right)^2 k^2 > 0$$

Case 2→

$$\frac{N^2}{U^2} - \alpha^2 < 0$$

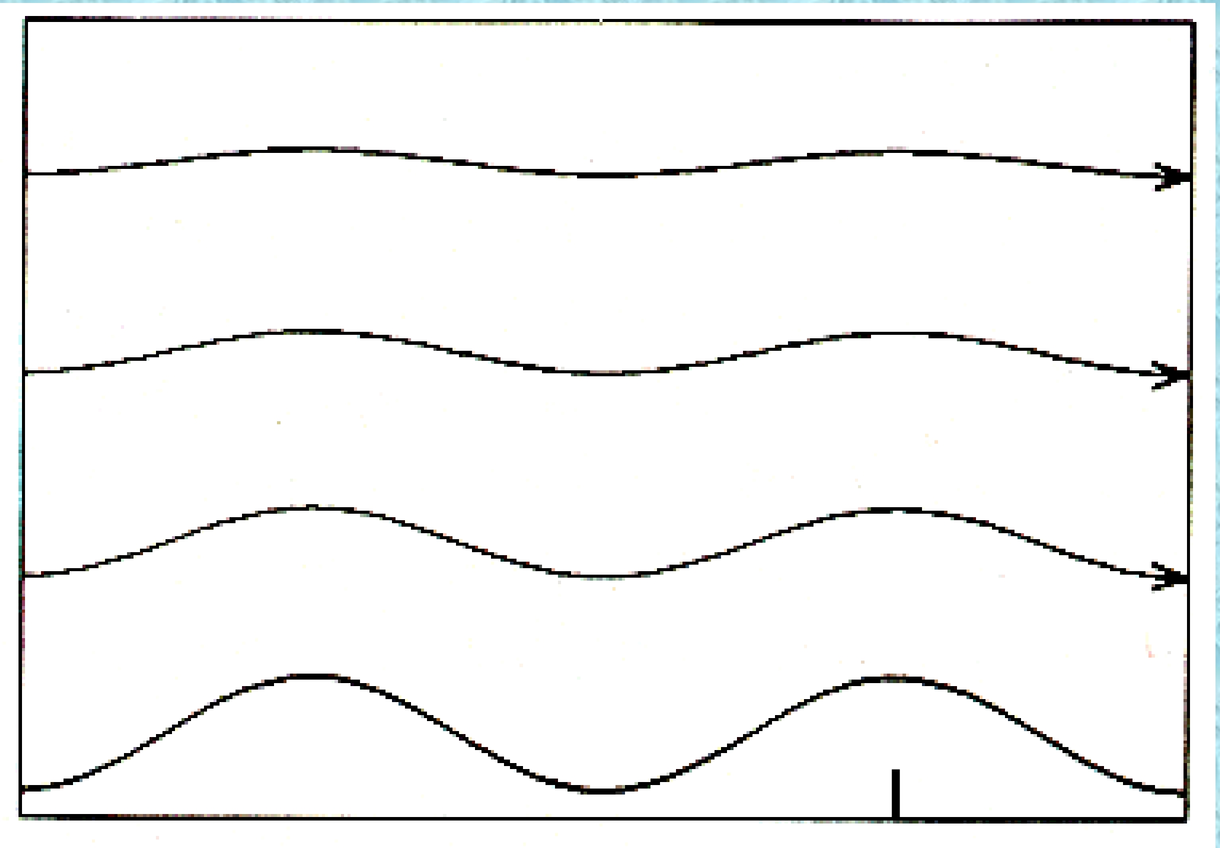
Sol'n →

$$\widehat{\psi}_+(z) = -\tfrac{1}{2} U H e^{\pm \mu z} \;,\; \widehat{\psi}_-(z) = -\tfrac{1}{2} U H e^{\pm \mu z}$$

$$\mu \equiv \left(\alpha^2 - \frac{N^2}{U^2} \right)^{\frac{1}{2}} > 0$$

use decay condition to indicate which sign to choose in the exponents

$$\psi(x,z) = -UHe^{-\mu z}\cos kx$$



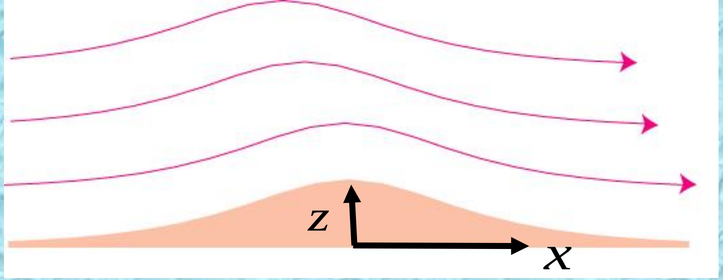
$x = 0$

$$(Uk)^2 > N^2$$

Doppler-shifted>Natural Freq

No Waves

Steady-State Linear Theory: Isolated Hill



$$h(x) = H \frac{L^2}{L^2 + x^2}$$

$$\psi(x, z = 0) = -UH \frac{L^2}{L^2 + x^2}$$

Fourier Transform

$$\hat{\psi}(z;k) = \int_{-\infty}^{+\infty} \psi(z,x) e^{-ikx} dx$$

Fourier Transform Eqn→

$$\frac{\partial^2 \hat{\psi}}{\partial z^2} + \left(\frac{N^2}{U^2} - k^2 \right) \hat{\psi} = 0$$

F.T. of B. C.→

$$\hat{\psi}(0;k) = UH\pi L^{-1} \exp(-|k|L)$$

Sol'n

$$\hat{\psi}(z;k) = \hat{\psi}(0;k) e^{\pm i m(k) z}$$

$$m(k) = \pm \left(\frac{N^2}{U^2} - k^2 \right)^{\frac{1}{2}}$$

Inverse Fourier Transform

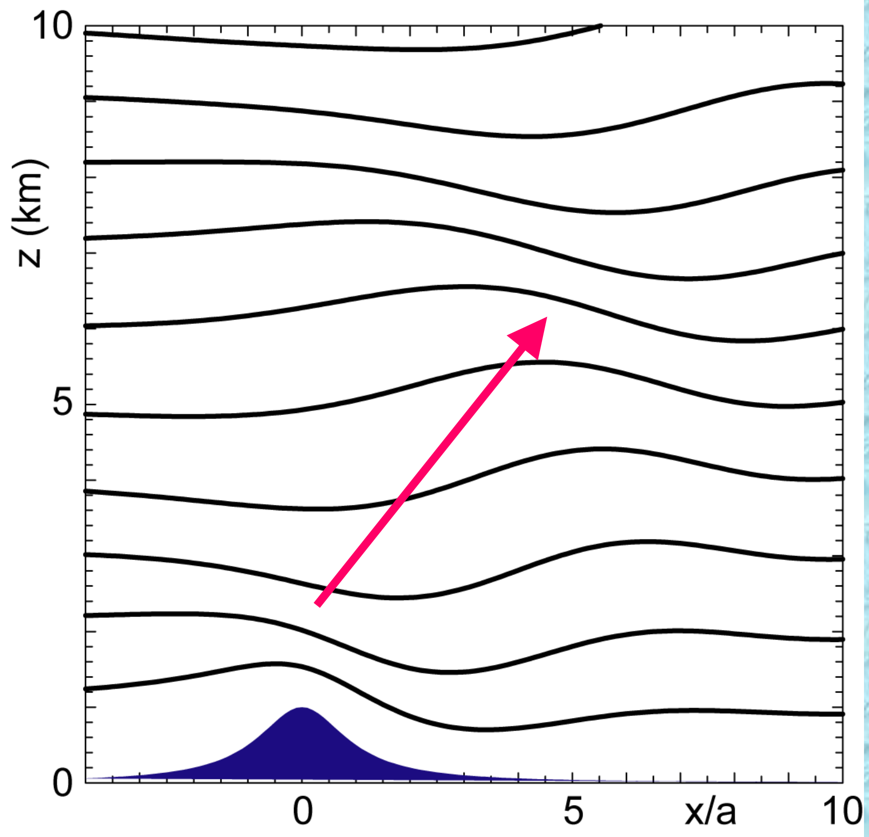
$$\psi(z,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\psi}(0;k) e^{i[kx \pm m(k)]z} dk$$

Radiation Condition for $m^2(k) > 0$

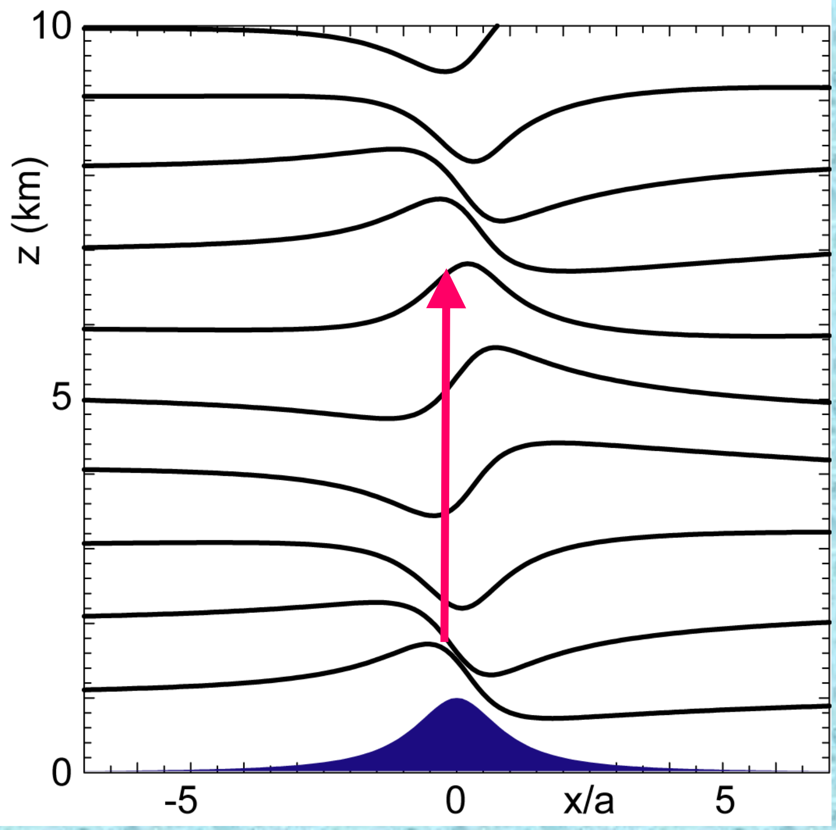
Decay Condition for $m^2(k) < 0$

$$\frac{c_{gx}}{c_{gz}} = \frac{k}{m} = \frac{k}{\sqrt{\frac{N^2}{U^2} - k^2}}$$

$$\frac{NL}{U} = 1$$



$$\frac{NL}{U} = 100$$



Summary

- Internal gravity waves are peculiar

  (upward energy but downward phase propagation)

- These basic ideas from IGW theory are necessary for the intepretation of mountian waves

- Literature is vast: more general situations such as *U=U(z)* , *N=N(z)* have been studied to understand flow response in particular atmospheric conditions