Higher-Order Finite-Volume Methods

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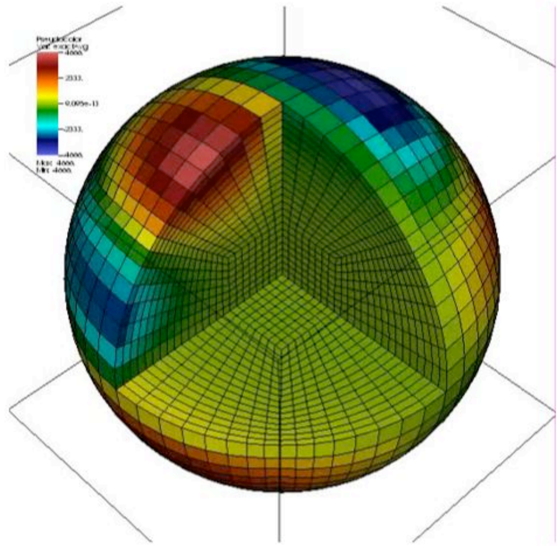
Mapped Multiblock Algorithms

Want to compute solutions to fluid dynamics problems in near-spherical symmetry without pole singularities.



Want to use finite-volume discretizations on multiblock grid obtained from cubed-sphere mapping. High-order accuracy (fourth-order or better) essential due to the discontinuities in the grid mapping at block boundaries.

Applications: Large-scale dynamics for supernovae; gyrokinetic edge plasmas in tokamaks; atmospheric fluid dynamics.

High-Order Finite-Volume Methods on Structured Grids

Local conservation form:

$$\nabla \cdot \vec{F} \rightarrow \frac{1}{h} \sum_{d=1}^{D} F^d_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} - F^d_{\boldsymbol{i}-\frac{1}{2}\boldsymbol{e}^d} \quad F^d_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} \approx \frac{1}{h^{D-1}} \int\limits_{A_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d}} F^d dA$$

Greater than second-order accuracy requires one to distinguish between point values and averages over control volumes, faces (Barad and Colella, 2006).

$$\frac{1}{h^{D-1}} \int\limits_{A_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d}} F^d dA = F^d(\boldsymbol{x}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d}) + \frac{h^2}{24} \sum_{d' \neq d} \frac{\partial^2 F^d}{\partial x^2_{d'}} + O(h^4)$$

Drivers:

- Applications requiring long-time integration.
- Lessens impact of loss of accuracy at boundaries where mesh is not smooth (e.g. AMR, multiblock): 2nd order $\rightarrow$ 1st order $\Rightarrow$ 4th order $\rightarrow$ 3rd order.
- Phase space problems (4-6 space dimensions).

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High-Order Finite-Volume Methods on Structured Grids

Extension to mapped grids:

$$\mathbf{x} = \mathbf{X}(\xi) \ , \ \mathbf{X} : [0,1]^D \rightarrow \mathbf{R}^D$$

$$\nabla_{\mathbf{x}} \cdot \vec{F} \equiv \frac{1}{J} \nabla_{\xi} \cdot (\mathbf{N}^T \vec{F})$$

$$J \equiv det(\nabla_{\xi} \mathbf{X}) \ , \ \mathbf{N}_{p,q} = det((\nabla_{\xi} \mathbf{X})(p|\mathbf{e}^q))$$

Finite-volume discretization: if $V_{\mathbf{i}}$ is a rectangular cell in the mapping space,

$$\int_{X(V_{\mathbf{i}})} \nabla_x \cdot \mathbf{F} d\mathbf{x} = \int_{V_{\mathbf{i}}} \nabla_{\xi} \cdot (\mathbf{N}^T \mathbf{F}) d\xi = \sum_{\pm = +,-} \sum_{d=1}^{D} \pm \int_{A_d^{\pm}} (\mathbf{N}^T \mathbf{F})_d dA_{\xi}$$

Fourth-order accurate approximation to face integrals:

$$\int_{A_d} (\mathbf{N}^T \vec{F})_d dA_{\xi} = \Big(\big(\int_{A_d} \mathbf{N}^T dA_{\xi} \big) \cdot \big(\int_{A_d} \vec{F} dA_{\xi} \big) \Big)_d + \frac{h^2}{12} \int_{A_d} \sum_{d' \neq d} \Big(\frac{\partial}{\partial \xi_{d'}} (\mathbf{N}^T) \cdot \frac{\partial}{\partial \xi_{d'}} (\vec{F}) \Big)_d dA_{\xi} + O(h^4)$$

Constant fluxes require cancellation of face integrals of metric terms.

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High-Order Finite-Volume Methods on Structured Grids

Freestream-preservation <-> equality of mixed partials:

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{N}^T = \sum_{\pm=+,-} \sum_{d=1}^{D} \pm \int_{A_{d,\pm}} \boldsymbol{N}_d^T dA_\xi = 0$$

Poincare lemma:

$$\exists \quad \mathcal{N}^s_{d,d'}, d \neq d' \quad \text{such that} \quad N^s_d = \sum_{d' \neq d} \frac{\partial \mathcal{N}^s_{d,d'}}{\partial \xi_{d'}} \; , \; \mathcal{N}^s_{d,d'} = -\mathcal{N}^s_{d',d}$$

Use Stokes' theorem to compute face averages of ***N***:

$$\int_{A_d} N^s_d dA_\xi = \sum_{\pm=+,-} \sum_{d' \neq d} \pm \int_{E^\pm_{d,d'}} \mathcal{N}^s_{d,d'} dE_\xi$$

Any quadrature for the integrals on the RHS that preserves antisymmetry implies that we get the required cancellation. $\mathcal{N}^s_{d,d'}$ is known as an explicit local function of the mapping and its gradient, for any number of dimensions.

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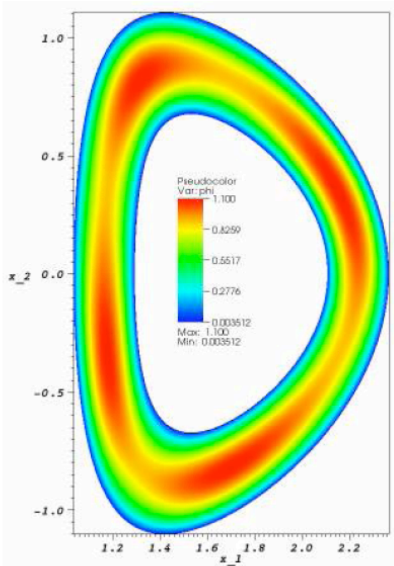
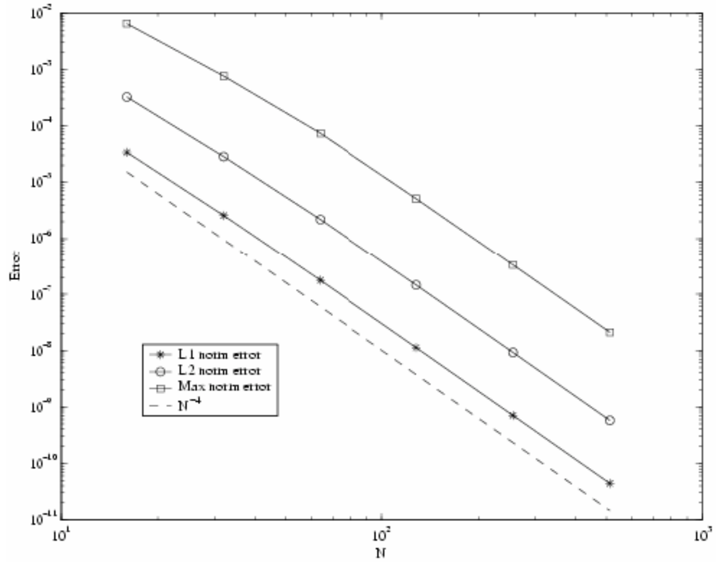
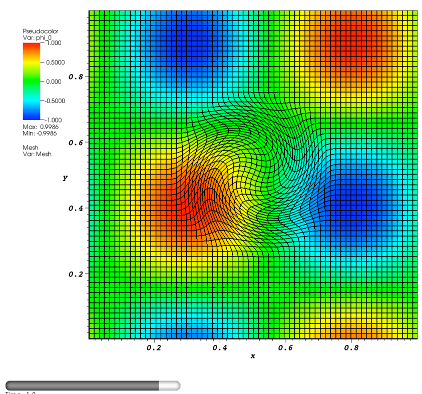
High-Order Finite-Volume Methods on Structured Grids

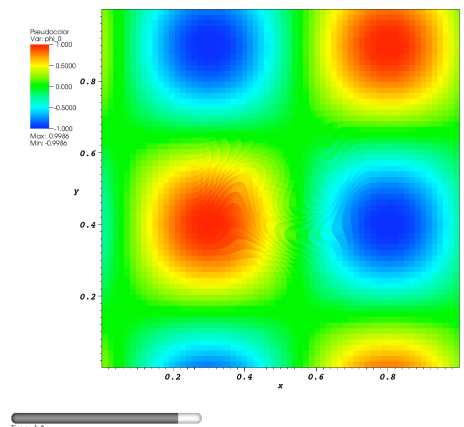
Gyrokinetic Poisson equation (variable-coefficient elliptic problem):





Advection on a twisted grid using centered differences, RK4 time discretization.





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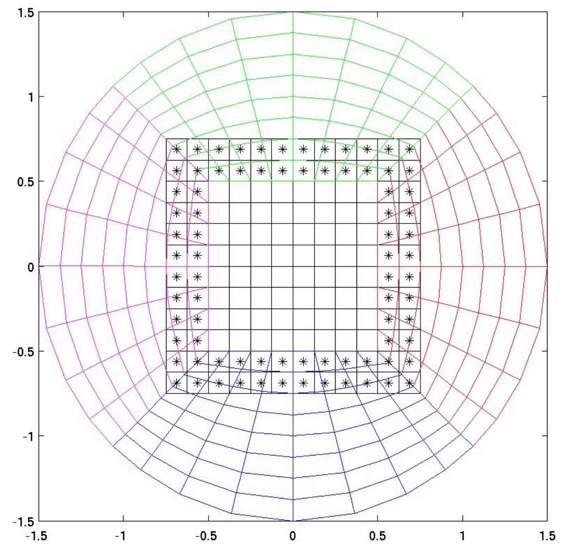
Mapped Multiblock Interpolation

Can extend higher-order mapped grid algorithms if one can compute sufficiently accurate ghost cell values on smooth extensions of each block.



Use polynomial interpolation:

$$\varphi(\boldsymbol{x}) \approx \sum_{\boldsymbol{p}} a_{\boldsymbol{p}} \boldsymbol{x}^{\boldsymbol{p}} \ , \ \boldsymbol{x}^{\boldsymbol{p}} = \prod_{d=1}^{D} x_d^{p_d}$$

$$\int_{V_{\boldsymbol{v}}} \varphi(\boldsymbol{x}) d\boldsymbol{x} = \sum_{\boldsymbol{p}} a_{\boldsymbol{p}} \int_{V_{\boldsymbol{v}}} \boldsymbol{x}^{\boldsymbol{p}} d\boldsymbol{x} \ , \ \boldsymbol{v} \in \mathcal{V}$$

We use a set of control volumes such that the above system of equations for the expansion coefficients is of maximal rank, and can be solved using least-squares. Then we evaluate the polynomial or its moments to obtain the ghost values.

High-Order Limiters (Colella and Sekora, JCP 2008)

Example: PPM (first step)

Linear deconvolution:

$$a_{j+\frac{1}{2}} = \frac{7}{12}(a_j + a_{j+1}) - \frac{1}{12}(a_{j-1} + a_{j+2})$$

Constrain face values so that

$$min(a_j, a_{j+1}) \leq a_{j+\frac{1}{2}} \leq max(a_j, a_{j+1})$$

At smooth extrema, this leads to a first-order accurate method. ENO / WENO / CENO, other methods provide remedies, but they are complicated - want a simpler solution.

High-Order Limiters (Colella and Sekora, JCP 2008)

At extrema (and only at extrema) replace the constraint

$$min(a_j, a_{j+1}) \leq a_{j+\frac{1}{2}} \leq max(a_j, a_{j+1})$$

with the following:

$$D^2 a = \frac{3}{h^2}(a_j + a_{j+1} - 2a_{j+\frac{1}{2}})$$

$$D^2 a_L = \frac{1}{h^2}(a_{j-1} + a_{j+1} - 2a_j)$$

$$D^2 a_R = \frac{1}{h^2}(a_j + a_{j+2} - 2a_{j+1})$$

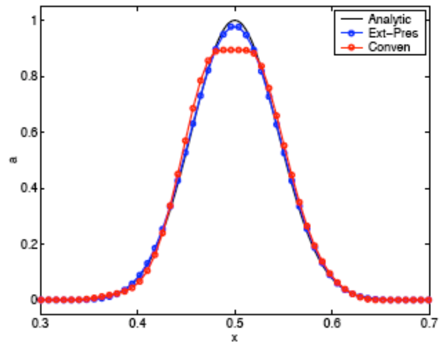
$$\begin{aligned} D^2 a_{lim} =& s \cdot min(C|D^2 a_L|, C|D^2 a_R|, |D^2 a|) \text{ if signs match} \\ =& 0 \text{ otherwise} \end{aligned}$$

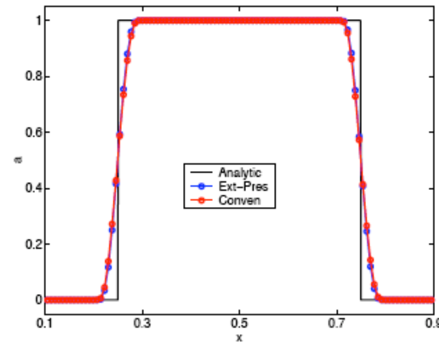
$$a_{j+\frac{1}{2}} = \frac{1}{2}(a_j + a_{j+1} - \frac{h^2}{3} D^2 a_{lim})$$

C > 1, independent of the mesh spacing. If the solution is smooth, edge value is unchanged. At a discontinuity, one of the estimates of D^2a is much smaller than the others. Can apply the same idea to limiting the parabolic profiles. Combine with Zalesak FCT limiter for positivity preservation.

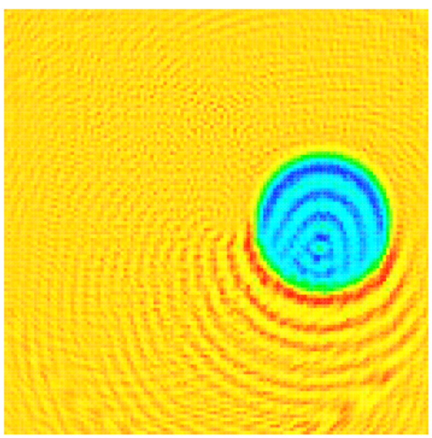
High-Order Limiters (Colella and Sekora, JCP 2008)

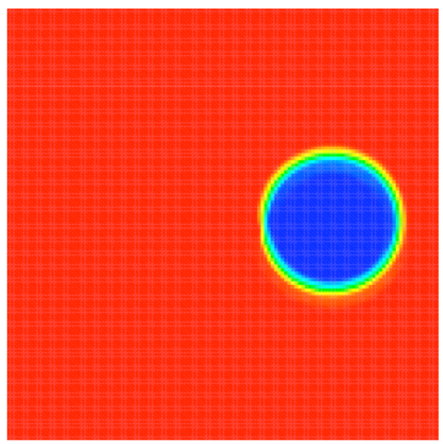
1D PPM advection results:

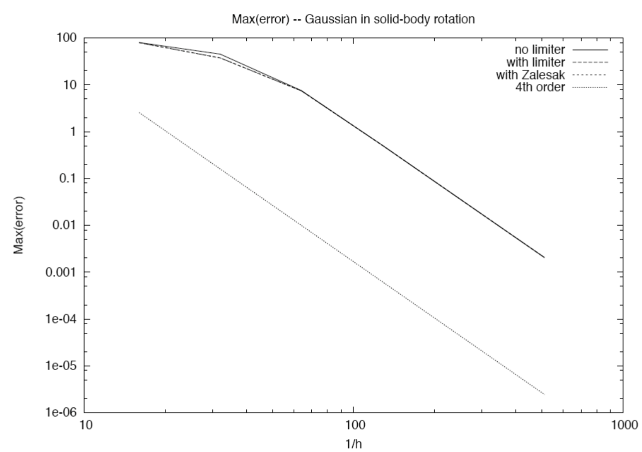




2D RK4 advection results (Colella, Dorr, Hittinger, Martin, 2008):







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Other Issues

Model issues. Splitting of three time scales (advective, gravity-wave, acoustic). Well-posed BVP for local refinement. Hydrostatic vs. non-hydrostatic behavior as a function of horizontal scale, within a non-hydrostatic model.

Other discretization Issues.

- High-order semi-implicit temporal discretization methods. Multiply-implicit spectral-deferred corrections in time, based on integral equation formulation ?

- Representation of orography: body-fitted grids vs. cut-cell.

Software Issues: retooling parallel AMR infrastructure for multiblock.

Planned Work

- Finish initial implementation of the mapped-multiblock infrastructure (less than six months).

- Advection, SWE on a sphere test cases (six months).

- Simplified dynamical core based on Helmholtz splitting of full compressible equations. No orography, column physics. Apply to test suite.

- Add column physics, for specialized two-scale simulation of the tropics ?