Galerken Methods for Global Atmospheric Dynamical Cores

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Galerkin Methods – Informal definition:

Expand unknowns in basis functions and exactly solve a system of integral equations

- Global Spectral Methods:
  - Basis functions are global and smooth (spherical harmonics)
- **Continuous Galerkin**
  - **AKA: the finite element method**
  - **Basis functions are globally C0, but have compact support over a few elements**
- Discontinuous Galerken (DG)
  - See Ram Nair's talk
  - Galerkin within each element, but elements tied together through edge/ surface fluxes

C0 Finite Element Method

Example equation for *h*: $$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0$$

Integral formulation: $$\int \psi \frac{\partial h}{\partial t} - \int \nabla \psi \cdot (h\boldsymbol{u}) = 0$$

Define a finite dimensional functional space, H1
(traditional choice: globally C0, piecewise polynomials)

Find $h \in H_1$ which solves equation *exactly* for all $\psi \in H_1$

Finite element method approximates the functional space, not the differential operators. This makes it easier to preserve many integral properties of these operators.

C0 Finite Element Method

Solve for h: $$\int \psi \frac{\partial h}{\partial t} - \int \nabla \psi \cdot (h\boldsymbol{u}) = 0 \quad \forall \psi \in H_1$$

Sufficient to find solution for every basis function: $\operatorname{span}\{\phi_i\} = H_1$

Expand h in this basis: $h = \sum_j h_j \phi_j$

$$\sum_j \int \phi_i \phi_j \frac{\partial h_j}{\partial t} = \int \nabla \phi_i \cdot (h\boldsymbol{u}) \qquad \forall \phi_i$$

$$\frac{\partial h_j}{\partial t} = M_{ij}^{-1} \int \nabla \phi_i \cdot (h\boldsymbol{u}) \qquad M_{ij} = \int \phi_i \phi_j$$

Mass Matrix

C0 Finite Element Method

- Using exact integration, mass matrix inversion means time dependent equations becomes implicit – not competitive?
- Juergen Steppeler: possible new approach for 2nd order formulation with diagonal mass matrix
- Spectral Element Method (implemented in NCAR's HOMME)
  - Diagonal Mass Matrix obtained by replacing integrals with GL quadrature and clever choice of basis functions.
  - Limited to quadrilateral or hexahedral grids. (no trianlges or hexagons)
  - Quadrature errors too large at low order (2, 3)
  - Expensive (CFL from GL points) at high order
  - 4th order (my personal preference)

Spectral Element Method

- Excellent Dynamics via *Compatibility*
  - Conserves anything in conservation form
  - Can conserve energy, vorticity in primitive variables
- Minimal Grid Imprinting
  - 4th order
  - FE method (treats all elements identically)
  - Hyperviscosity (grad^4) instead of grid-dependent limiters.
- Consistent Advection
  - No limiter: (4th order) oscillatory and not acceptable
  - Sign preserving + hyperviscosity (3rd order)
  - Monotone (2nd order) and more dissipative than FV/DG for advection of discontinuities
- AMR
  - One of the few methods where local mesh refinement actually reduces the global error levels (shallow water test cases on the sphere)
  - Conforming grids (Fournier et al, MWR 2004)
  - Nonconforming grids (St-Cyr et al, MWR 2006)

Compatible Numerical Methods
Local Properties

The key integral property of the continuum equations needed to show local conservation, for scalar h and vector v, is:

$$\int_{\Omega} \boldsymbol{v} \cdot \nabla h + \int_{\Omega} h \nabla \cdot \boldsymbol{v} = \oint_{\partial\Omega} h \boldsymbol{v} \cdot \hat{n}$$

Taking Ω to be a single element, *the spectral element gradient and divergence operators DIV() and GRAD() satisfy:*

$$\sum_{\Omega} \boldsymbol{v} \cdot \mathrm{GRAD}(h) + \sum_{\Omega} h \, \mathrm{DIV}(\boldsymbol{v}) = \sum_{\partial\Omega} h \boldsymbol{v} \cdot \hat{n}$$

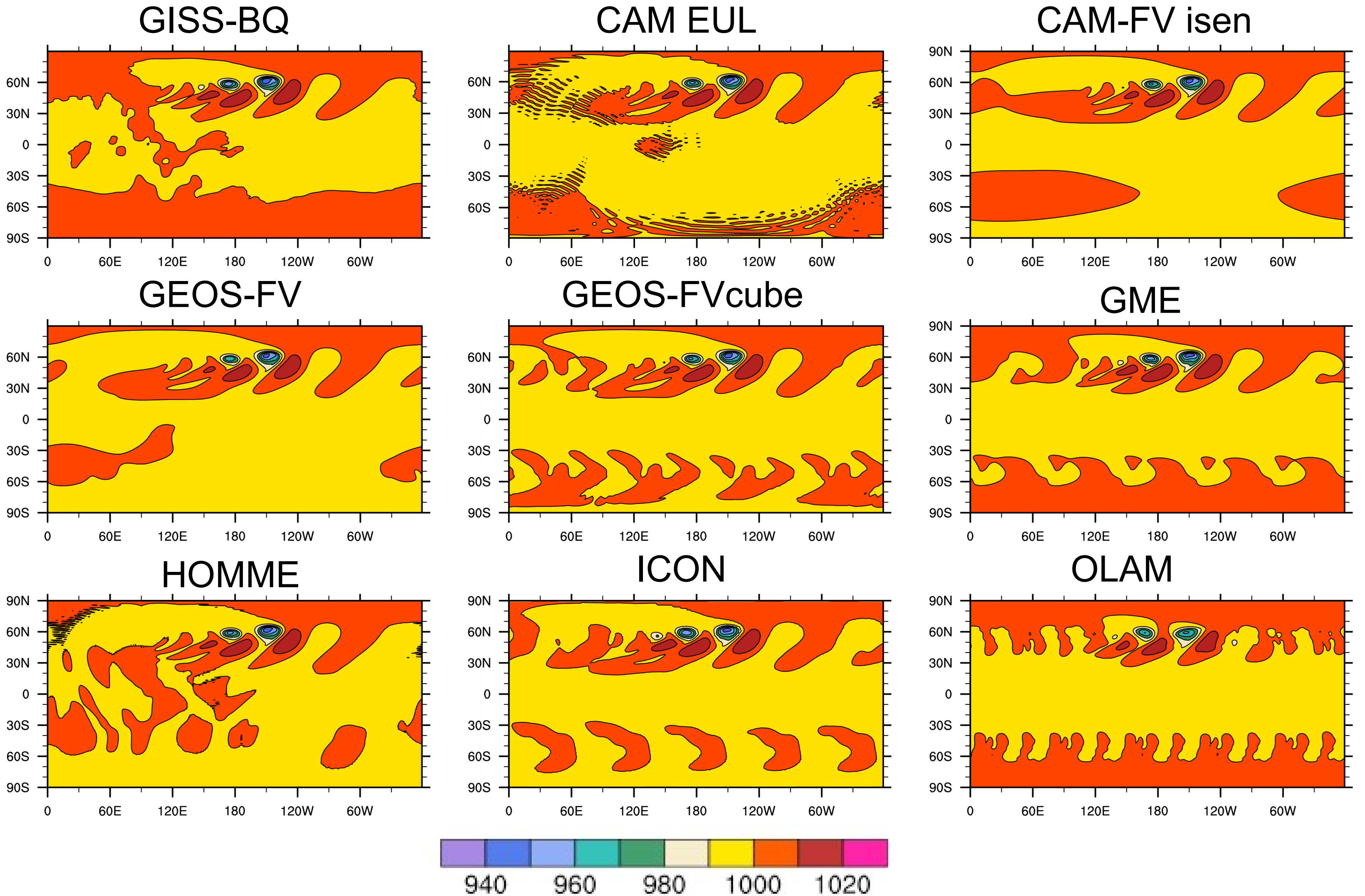
Where the sum over Ω is the Gauss-Lobatto approximation to the integral over an element, and the sum over the boundary of Ω is the natural Gauss-Lobatto approximation to the line integral around the boundary of the element.

Compatible Numerical Methods

Discrete operators and discrete integral satisfy continuum properties:

$$\int \nabla \cdot (p\boldsymbol{v}) = \int p \nabla \cdot \boldsymbol{v} + \int \boldsymbol{v} \cdot \nabla p = 0$$
$$\int \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \int \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \int \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} = 0$$
$$\nabla \times \nabla p = 0$$
$$\nabla \cdot \nabla \times \boldsymbol{u} = 0$$

- Integration by parts insures conservation
- Curl Grad = 0 can improve vorticity evolution
- Many schemes have this property on orthogonal Cartesian grids
- Continuous Galerkin methods have these properties on arbitrary grids in general curvilinear coordinates.



Test 2: Baroclinic instability. Surface pressure at day 9. The tests starts with balanced initial conditions that are overlaid by a Gaussian hill perturbation. The perturbation grows into a baroclinic wave. Some models show cubed-sphere or icosahedral grid imprinting in the Southern Hemisphere. High order methods show spectral ringing in the 1000mb contour.

Energy Balance in Aqua Planet
CAM/HOMME moist hydrostatic primitive equations

- **Dissipation: Aqua planet simulations need about 1 W/m^2 KE dissipation**
  - Far more than is needed to control 2 dx mode.
  - Dissipation too closely tied to the grid can lead to large grid imprinting (replaced element filters with hyperviscosity)
  - KE dissipation added to T equation (à la CAM-EUL)
  - Remaining TE dissipation is from Robert filter and Q dissipation that is not added to T equation

Example: from a typical snapshot in Aqua Planet:
KE = 0.28e7 J/m^2
IE = 0.26e10 J/m^2

	Forcing	Transfer	Dissipation	(W/m^2)
d(KE)/dt =	-2.6	+2.8	-0.86	
d(IE)/dt =	0.83	-2.8	0.86	

TE Numerical Diffusion: -0.00061 W/m^2

Challenges for a Non-Hydrostatic version of HOMME-SE

- Collocated method: has A-grid like 2 dx mode that *requires* dissipation (KE or tracer variance).
- If limiters are needed on *density:*
  - Exact conservation only of quantities in conservation form
  - If quasi-monotone advection is required, method drops to 2nd order. Can we come up with a 3rd order quasi-monotone limiter?
- Vertical Coordinates
  - 2D + Lagrange?
  - 3D spectral elements?
- Non-hydrostatic equation formulation
  - Conservation form and some primitive variable formulations
- We are still using leapfrog + Robert filter

Spectral Element Advection Slides

Sign Preserving and Monotone Advection

Advection Equation $$\frac{\partial}{\partial t}(\rho q) = -\nabla \cdot (\rho q)\boldsymbol{u}$$

Spectral Elements: $$\frac{\partial}{\partial t}(\rho q)_j = M_{ij}^{-1} \int \nabla \phi_i \cdot (\rho q \boldsymbol{u})$$

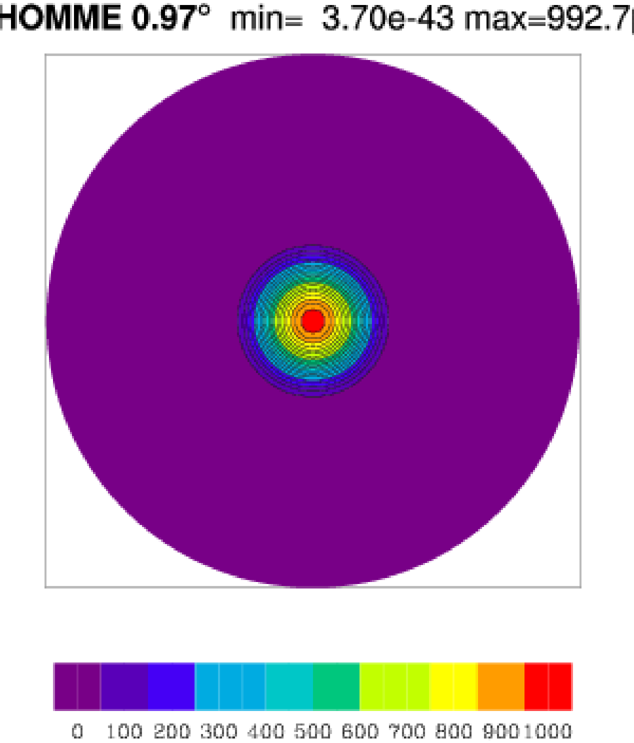
Apply Leapfrog: $$(\rho q)^{t+1} = (\rho q)^{t-1} + 2\Delta t\, M^{-1}\, RHS(t)$$

Equivalent to: $$(\rho q)^{*} = (\rho q)^{t-1} + 2\Delta t\, RHS(t)$$
$$(\rho q)^{t+1} = M^{-1} (\rho q)^{*}$$

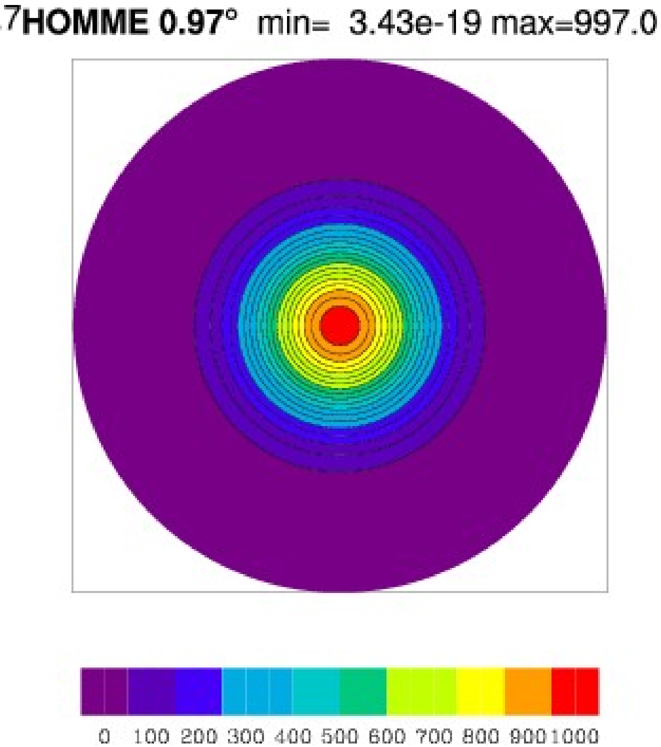
Theorem: The element means q_0, defined so that the element mass $(\rho q)^* = (\rho q_0)^*$, is monotone. Thus it is always possible to find a mass conserving monotone reconstruction of q^* within each element, *before* application of M^{-1}

The spectral element mass matrix inverse will preserve monotonicity. (not true for general CG methods)

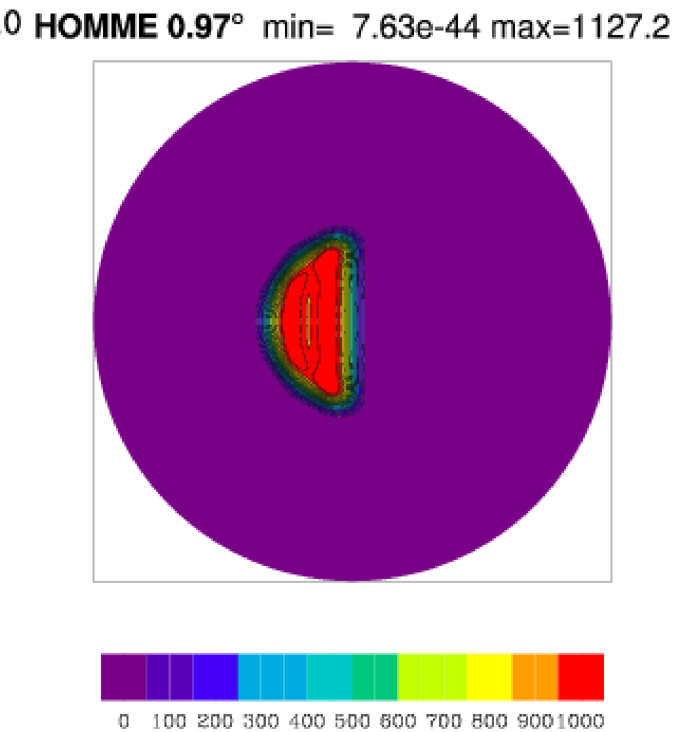
Cosine Bell

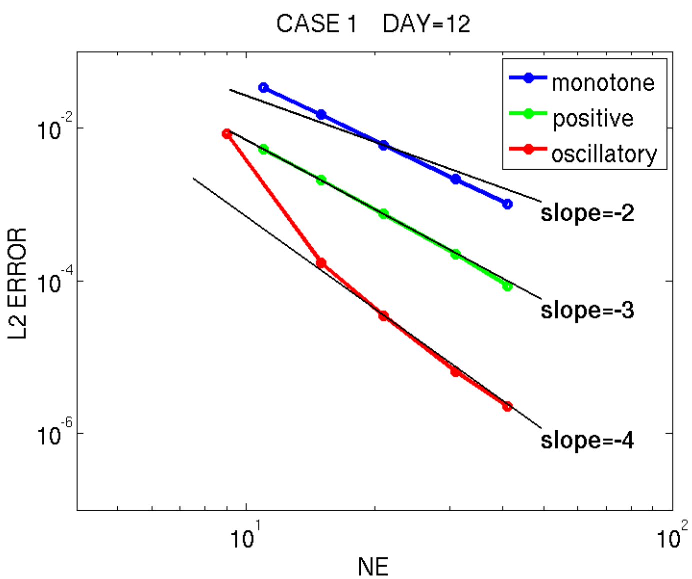


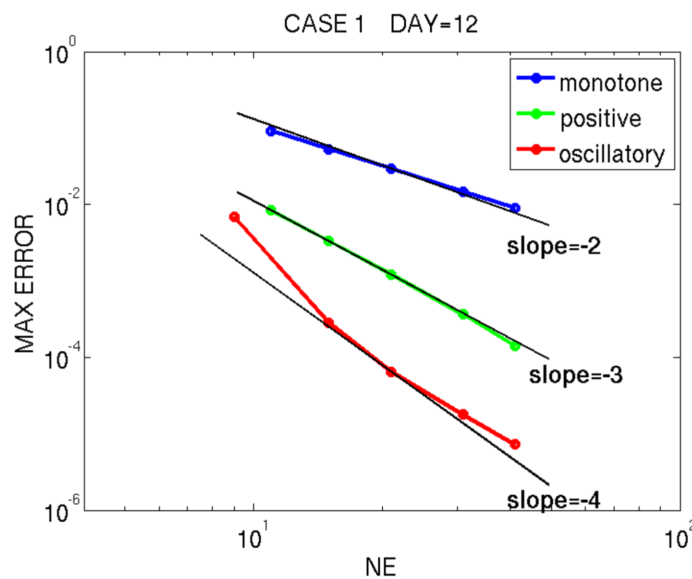
Gaussian



Half Cylinder





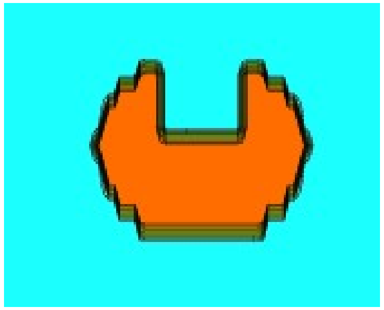


Sign Preserving and Monotone Advection
Primitive Equations

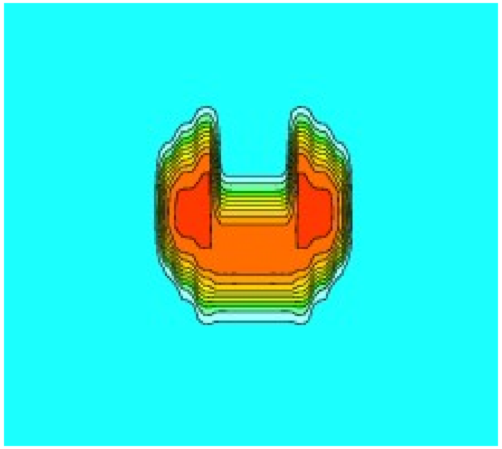
2D spectral element advection coupled with Lagrangian vertical coordinate and Lauritzen PPM based remap.

Remap appied twice per timestep (because of Leapfrog).

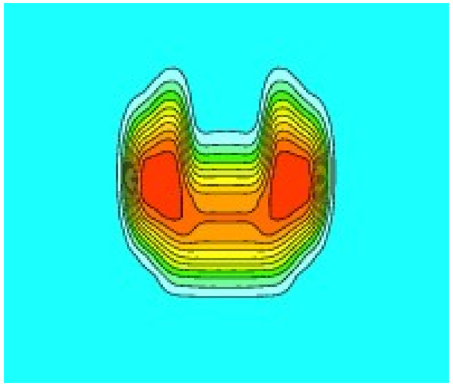
Initial Condition

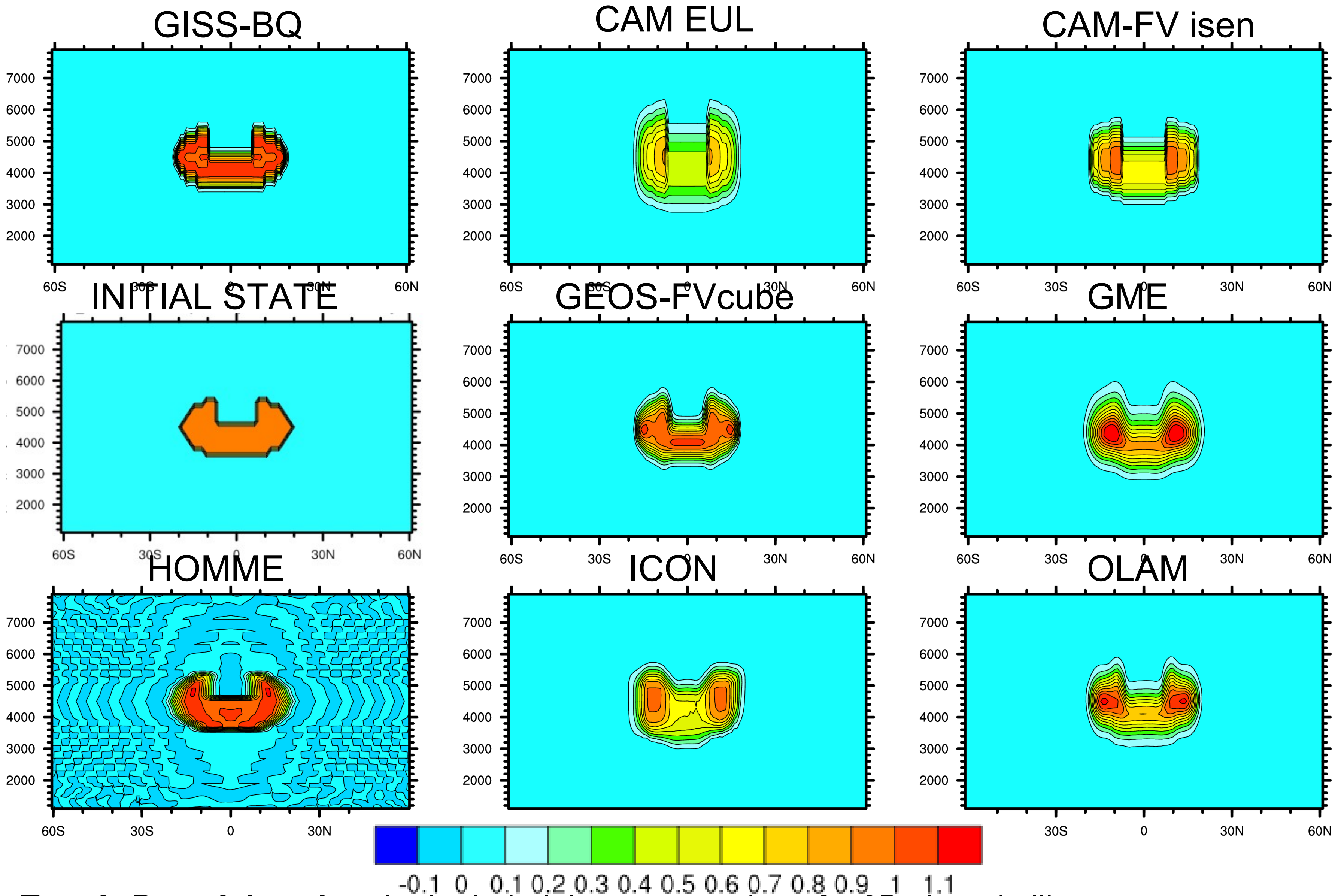


GEOS-FV



HOMME





Test 3: Pure Advection. Latitude-height cross section of a 3D slotted ellipse tracer distribution after one revolution around the sphere (day 12). The 3D winds are prescribed. The slotted ellipse has followed a trajectory path with three wave cycles in the vertical direction. The test evaluates the diffusion characteristics of the advection algorithm.

Backup Slides

Hyper Viscosity: mixed FE formulation

$$\frac{\partial h}{\partial t} + \ \ldots \ = -\nu \Delta b$$

$$b = \Delta h$$

Weak form, integrated by parts:

$$\int \psi \frac{\partial h}{\partial t} + \ \ldots \ = -\nu \int \nabla \psi \cdot \nabla b$$

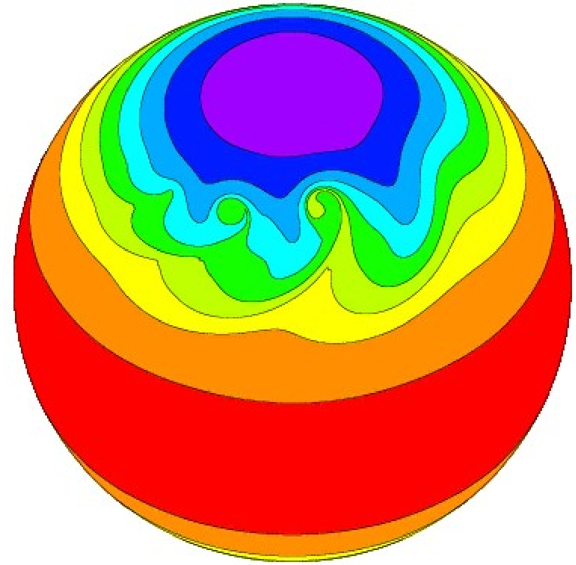
$$\int \psi b = \int \nabla \psi \cdot \nabla h$$

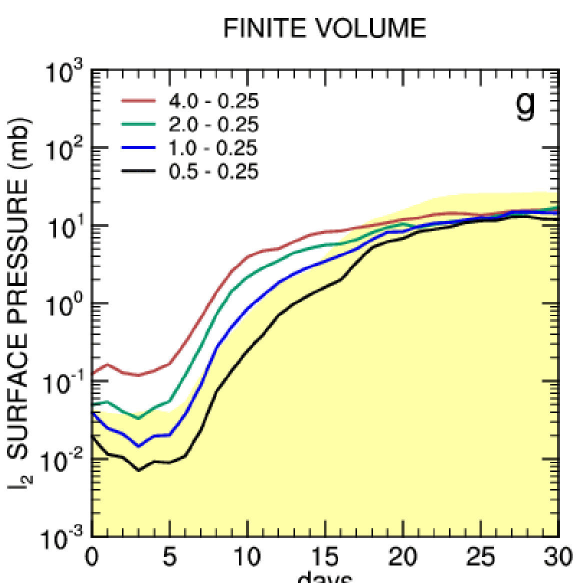
F. Giraldo, *Trajectory Calculations for Spherical Geodesic Grids in Cartesian Space,* MWR 1999

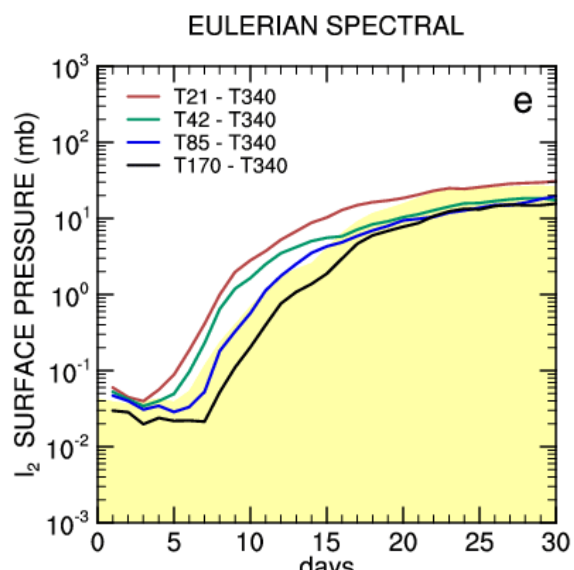
Test 2: Baroclinic Instability Test

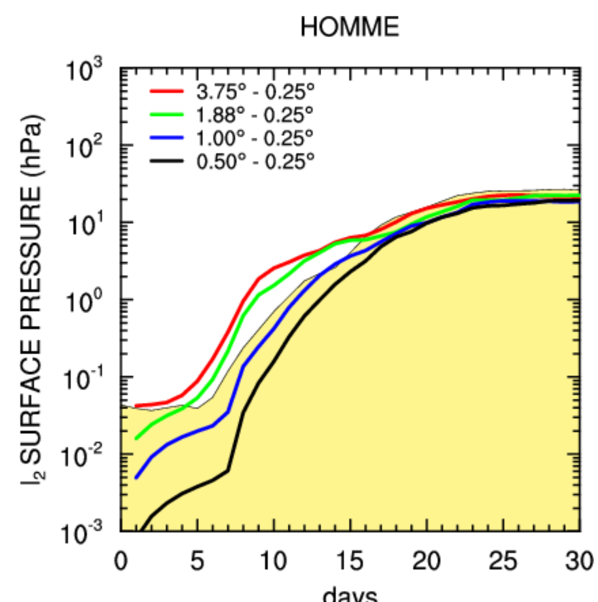
Jablonowski and Williamson, *A Baroclinic Instability Test Case for Atmospheric Model Dynamical Cores*, Q.J.R. Meteorol. Soc. (2006)

- Dynamical core only: no atmospheric physics
- L2 error in surface pressure as a function of time shown below
- Converges under mesh refinement to reference solution (uncertainty in reference solution is yellow shaded region)

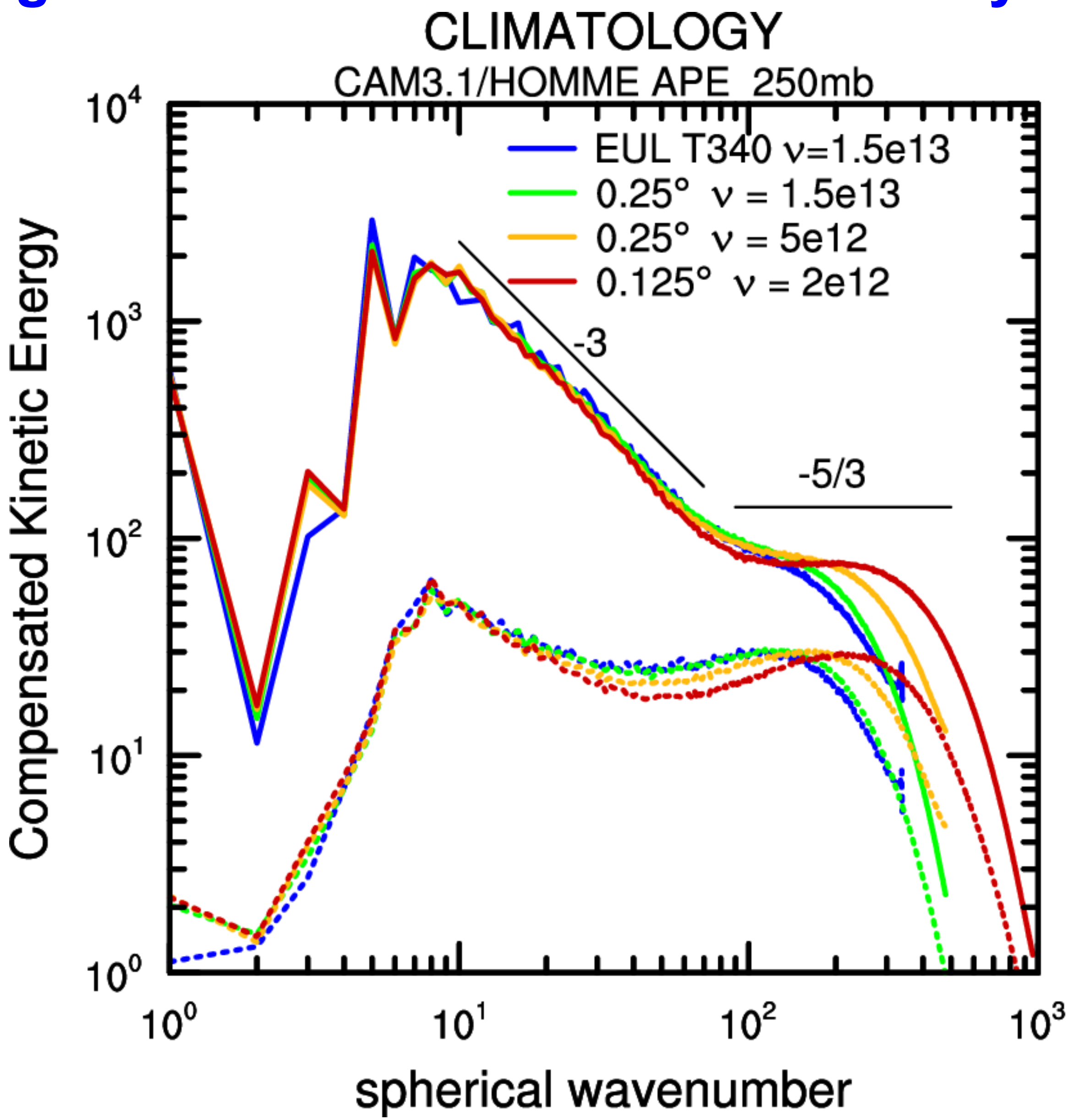








High Resolution Results - CAM 3.4 Physics



Aqua Planet Global Mean Quantities

Resolution	Physics dt	Viscosity	PRECC	PRECL	CLDTOT	TMQ
EUL T42	5m	1.0E+16	1.71	1.11	0.64	20.21
HOMME 1.9	5m	1.0E+16	1.76	1.14	0.66	20.09
EUL T85	5m	1.0E+15	1.59	1.38	0.60	19.63
HOMME 1.0	5.5m	1.0E+15	1.59	1.43	0.61	19.67
HOMME 1.0	5.5m	3.0E+14	1.45	1.58	0.59	19.71
EUL T170	5m	1.5E+14	1.44	1.62	0.55	19.13
HOMME 0.5	5m	1.5E+14	1.48	1.62	0.55	19.36
HOMME 0.5	5m	5.0E+13	1.39	1.70	0.53	19.18
T340	5m	1.5E+13	1.36	1.75	0.50	18.75

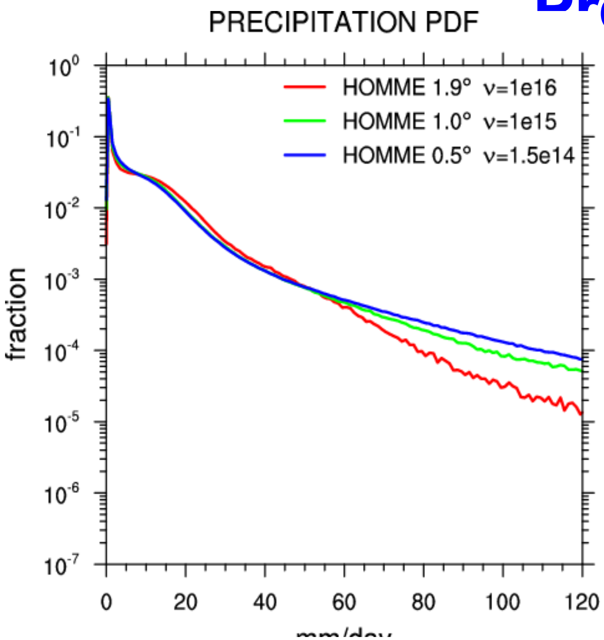
Compared to the size of the resolution signal, there is a remarkable agreement between CAM/HOMME and CAM/Eulerian

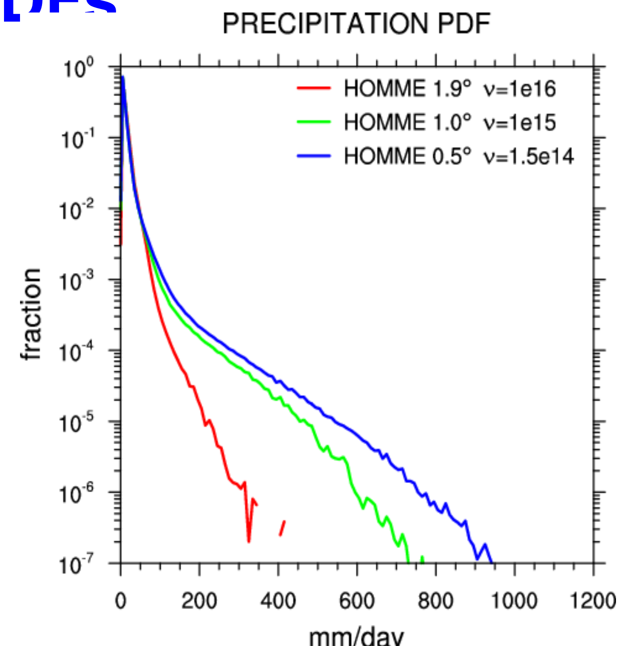
Precipitation PDFs

1mm bin-size

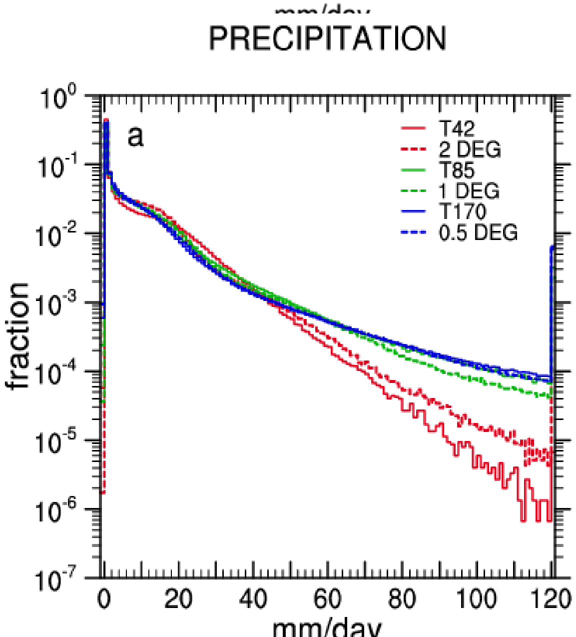
10mm bin-size

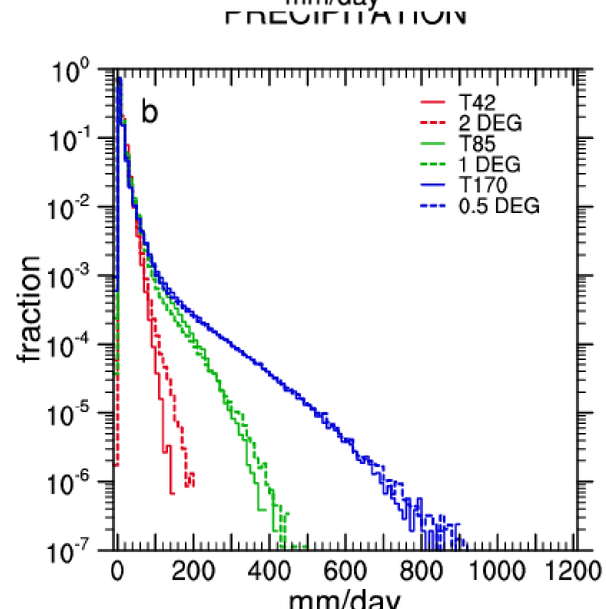
HOMME



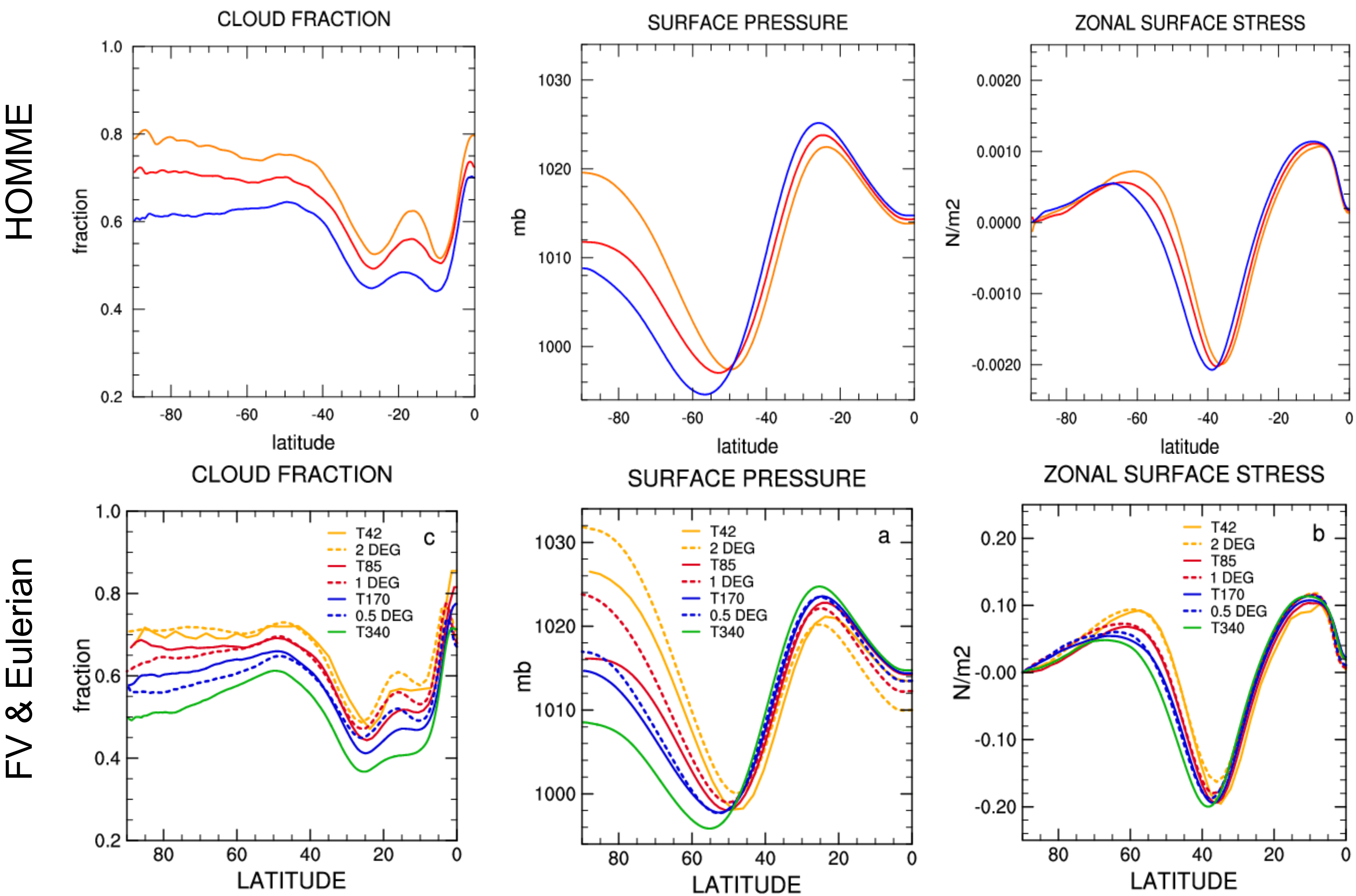


FV & Eulerian

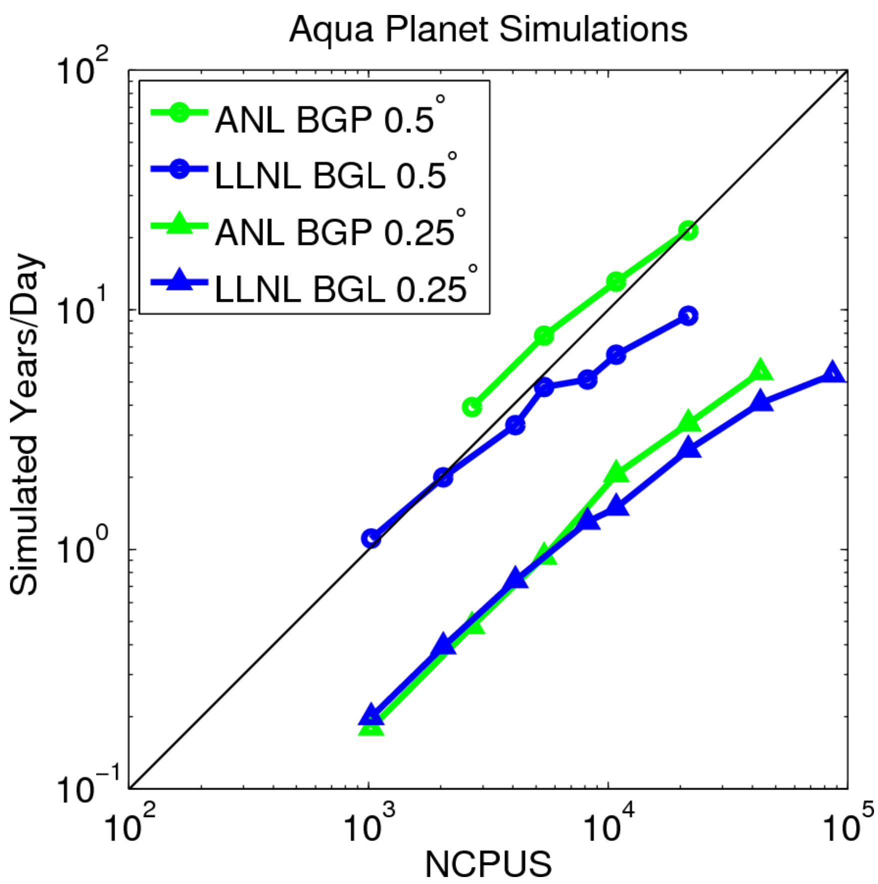




Aqua Planet Experiment: Zonal Data Comparison with FV & Eulerian Dycore



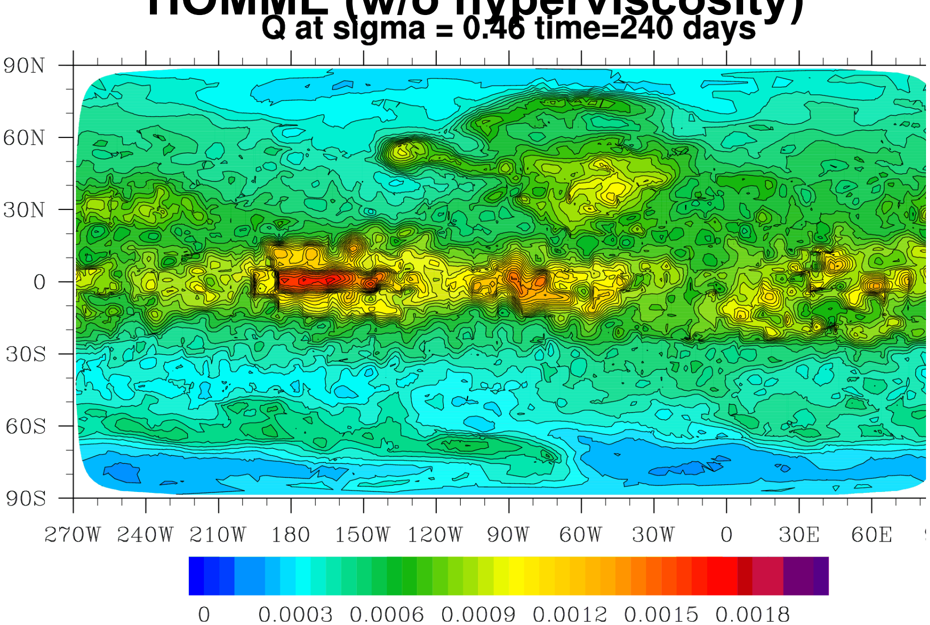
Fixed Mesh Scalability CAM/HOMME



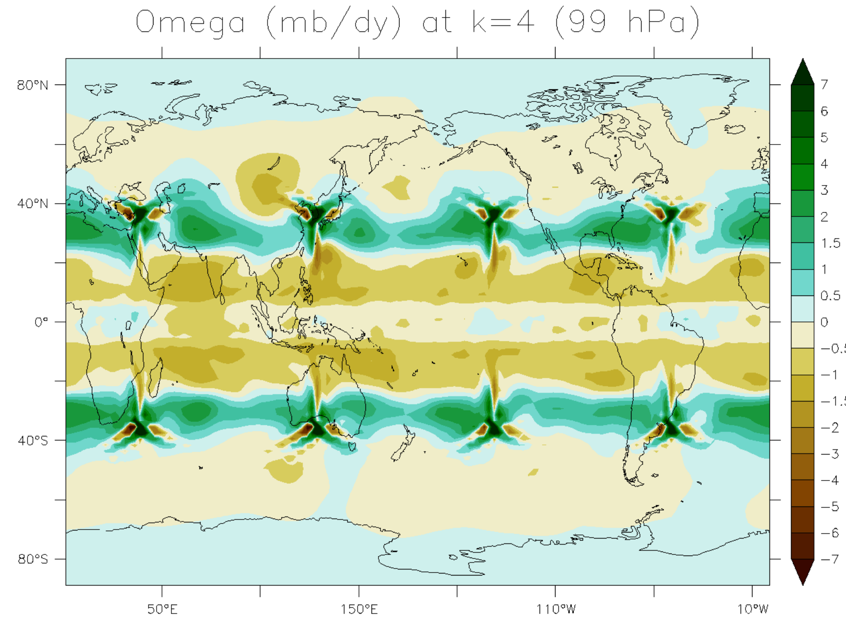
- Good scalability down to 1 element per processor for both resolutions, suggesting target 0.1 degree resolution should scale to 250K processors.
- Integration rates better than 5 simulated years/day at resolutions down to 0.25 degree
- BGL results: 1 processor per node due to memory constraints. BGP results use 4 processor cores per node. BGP is 4x-8x faster per node.

Gallery of Cubed-Sphere Problems

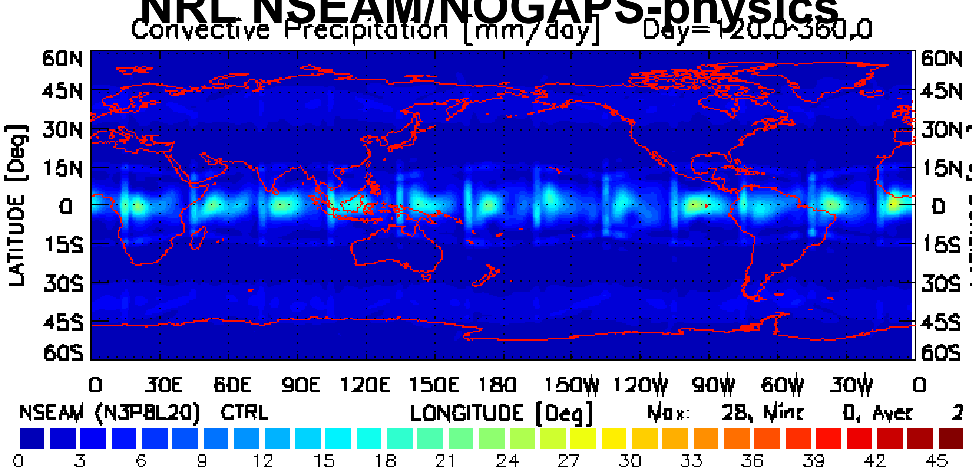
HOMME (w/o hyperviscosity)



NASA/GFDL FV Model



NRL NSEAM/NOGAPS-physics



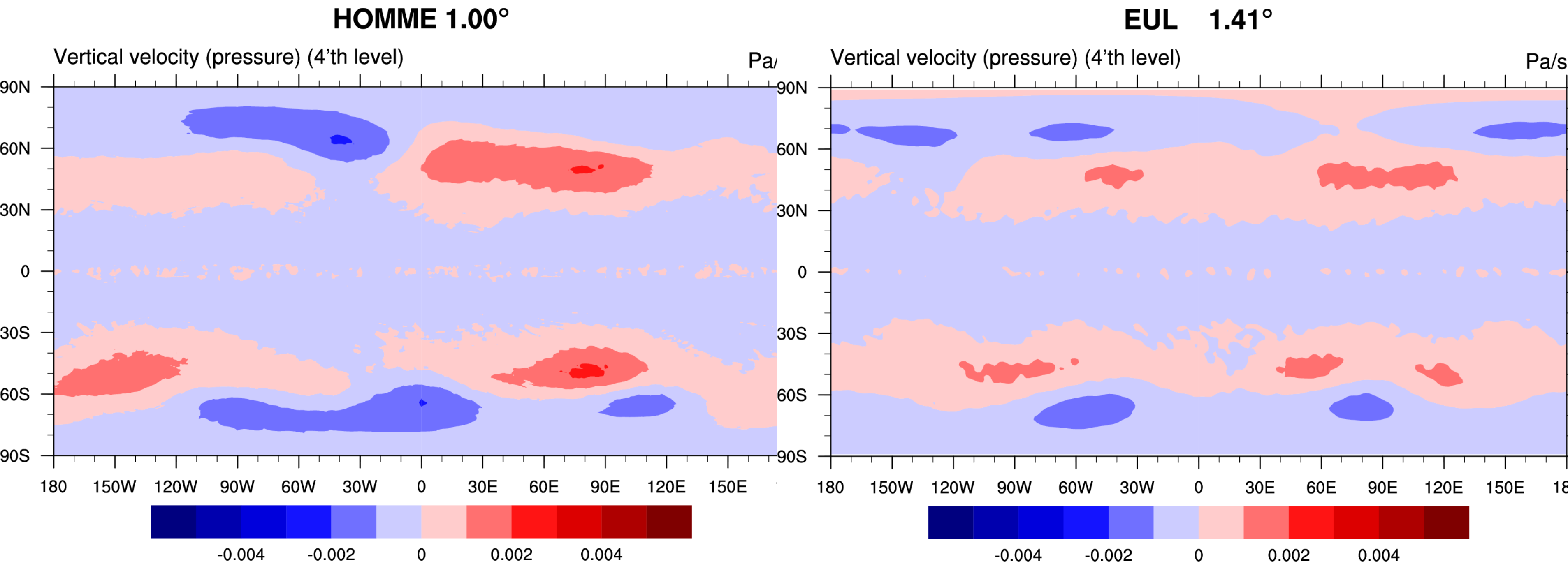
Sources:

M Taylor et al., 2007 PDEs on the Sphere Workshop, Exeter

YJ Kim et al., 2006 PDEs on the Sphere Workshop, Monterey

B. Wyman et al., 12th Annual CCSM Workshop, June 19-21, 2007

Minimal cubed-sphere grid imprinting



Pressure vertical velocity contoured on the 4'th eta-level. This field is one of the most sensitive to grid imprinting.

Noise characteristics of CAM/HOMME quite similar to the near perfectly isotropic CAM/Eulerian model.