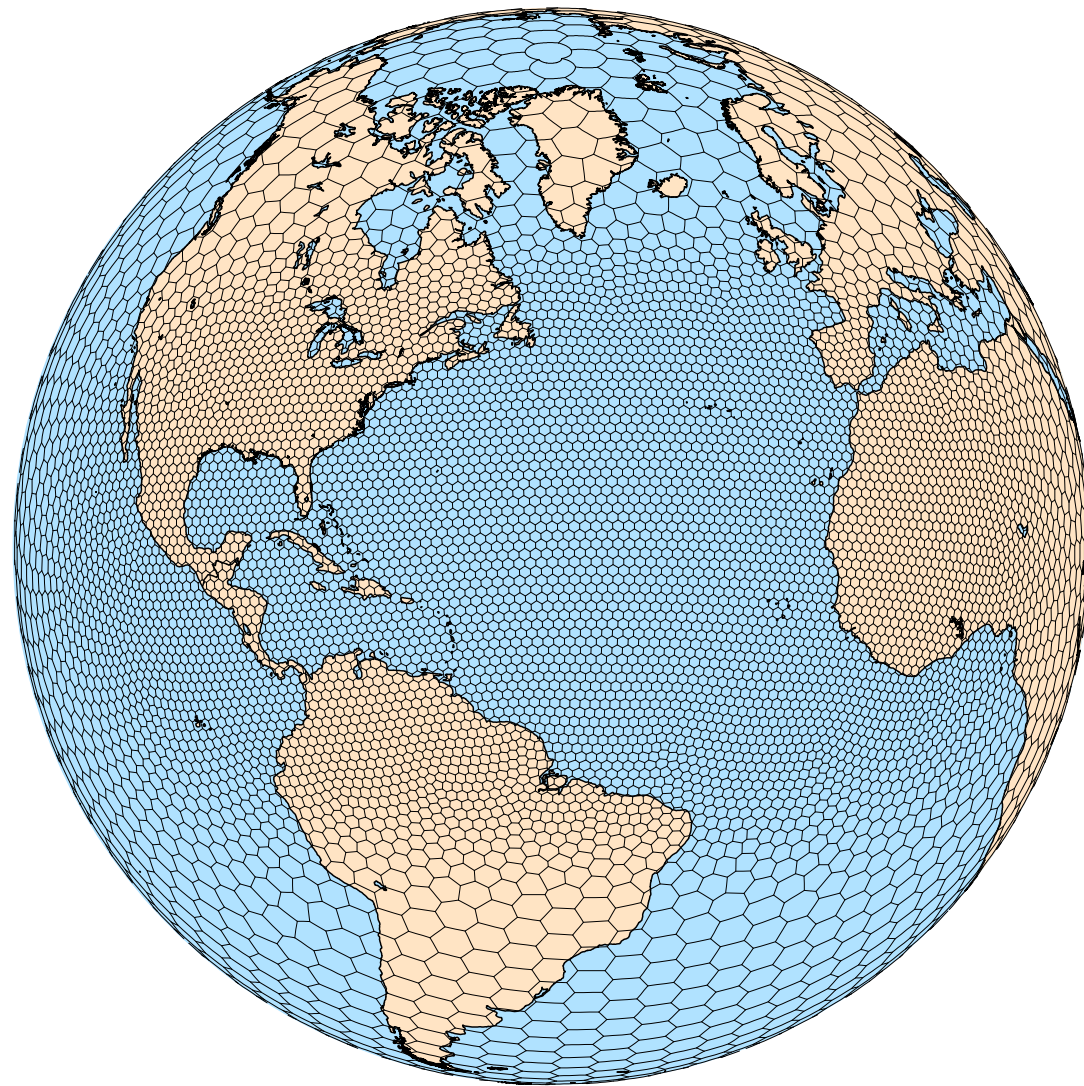


Dynamical Core

- *Time integration*
 - *Algorithms*
 - *Timesteps*
 - *Namelist parameters*
 - *References*
- *Spatial Discretization for the dynamics*



MPAS Nonhydrostatic Atmospheric Solver

Equations

- Prognostic equations for coupled variables.
- Generalized height coordinate.
- Horizontally vector-invariant equation set.
- Continuity equation for dry air mass.
- Thermodynamic equation for coupled potential temperature.

Variables: $(U, V, \Omega, \Theta, Q_j) = \tilde{\rho}_d (u, v, \omega, \theta, q_j)$ $\tilde{\rho}_d = \rho_d / \zeta_z$

Vertical coordinate: $z = \zeta + A(\zeta)h_s(x, y, \zeta)$

Prognostic equations:

$$\begin{aligned} \frac{\partial \mathbf{V}_H}{\partial t} &= -\frac{\rho_d}{\rho_m} \left[\nabla_\zeta \left(\frac{p}{\zeta_z} \right) - \frac{\partial \mathbf{z}_H p}{\partial \zeta} \right] - \eta \mathbf{k} \times \mathbf{V}_H \\ &\quad - \mathbf{v}_H \nabla_\zeta \cdot \mathbf{V} - \frac{\partial \Omega \mathbf{v}_H}{\partial \zeta} - \rho_d \nabla_\zeta K + \mathbf{F}_{V_H} \\ \frac{\partial W}{\partial t} &= -\frac{\rho_d}{\rho_m} \left[\frac{\partial p}{\partial \zeta} + g \tilde{\rho}_m \right] - (\nabla \cdot \mathbf{v} W)_\zeta + F_W \\ \frac{\partial \Theta_m}{\partial t} &= -(\nabla \cdot \mathbf{V} \theta_m)_\zeta + F_{\Theta_m} \\ \frac{\partial \tilde{\rho}_d}{\partial t} &= -(\nabla \cdot \mathbf{V})_\zeta \\ \frac{\partial Q_j}{\partial t} &= -(\nabla \cdot \mathbf{V} q_j)_\zeta + F_{Q_j} \end{aligned}$$

Diagnostics and definitions:

$$\frac{\rho_m}{\rho_d} = 1 + q_v + q_c + q_r + \dots$$

$$p = p_0 \left(\frac{R_d \zeta_z \Theta_m}{p_0} \right)^\gamma \quad \theta_m = \theta [1 + (R_v/R_d)q_v]$$

Time Integration

3rd Order Runge-Kutta time integration

Advance one
time step $\phi^t \rightarrow \phi^{t+\Delta t}$

$$\frac{\partial U}{\partial t} = RHS_u$$

$$\frac{\partial W}{\partial t} = RHS_w$$

$$\phi^* = \phi^t + \frac{\Delta t}{3} RHS(\phi^t)$$

$$\phi^{**} = \phi^t + \frac{\Delta t}{2} RHS(\phi^*)$$

$$\phi^{t+\Delta t} = \phi^t + \Delta t RHS(\phi^{**})$$

-
-
-

$$\phi_t = i k \phi; \quad \phi^{n+1} = A \phi^n; \quad |A| = 1 - \frac{(k \Delta t)^4}{24} + \text{H.O.T}$$

Time Integration

2nd-order RK variant – default in MPAS

Advance one time step $\phi^t \rightarrow \phi^{t+\Delta t}$

$$\frac{\partial U}{\partial t} = RHS_u$$

$$\frac{\partial W}{\partial t} = RHS_w$$

$$\phi^* = \phi^t + \frac{\Delta t}{2} RHS(\phi^t)$$

$$\phi^{**} = \phi^t + \frac{\Delta t}{2} RHS(\phi^*)$$

$$\phi^{t+\Delta t} = \phi^t + \Delta t RHS(\phi^{**})$$

-
-
-

$$\phi_t = i k \phi; \quad \phi^{n+1} = A \phi^n; \quad |A| = 1 - \frac{(k \Delta t)^3}{12} + \text{H.O.T}$$

$$\phi_t = i k \phi; \quad \phi^{n+1} = A \phi^n$$

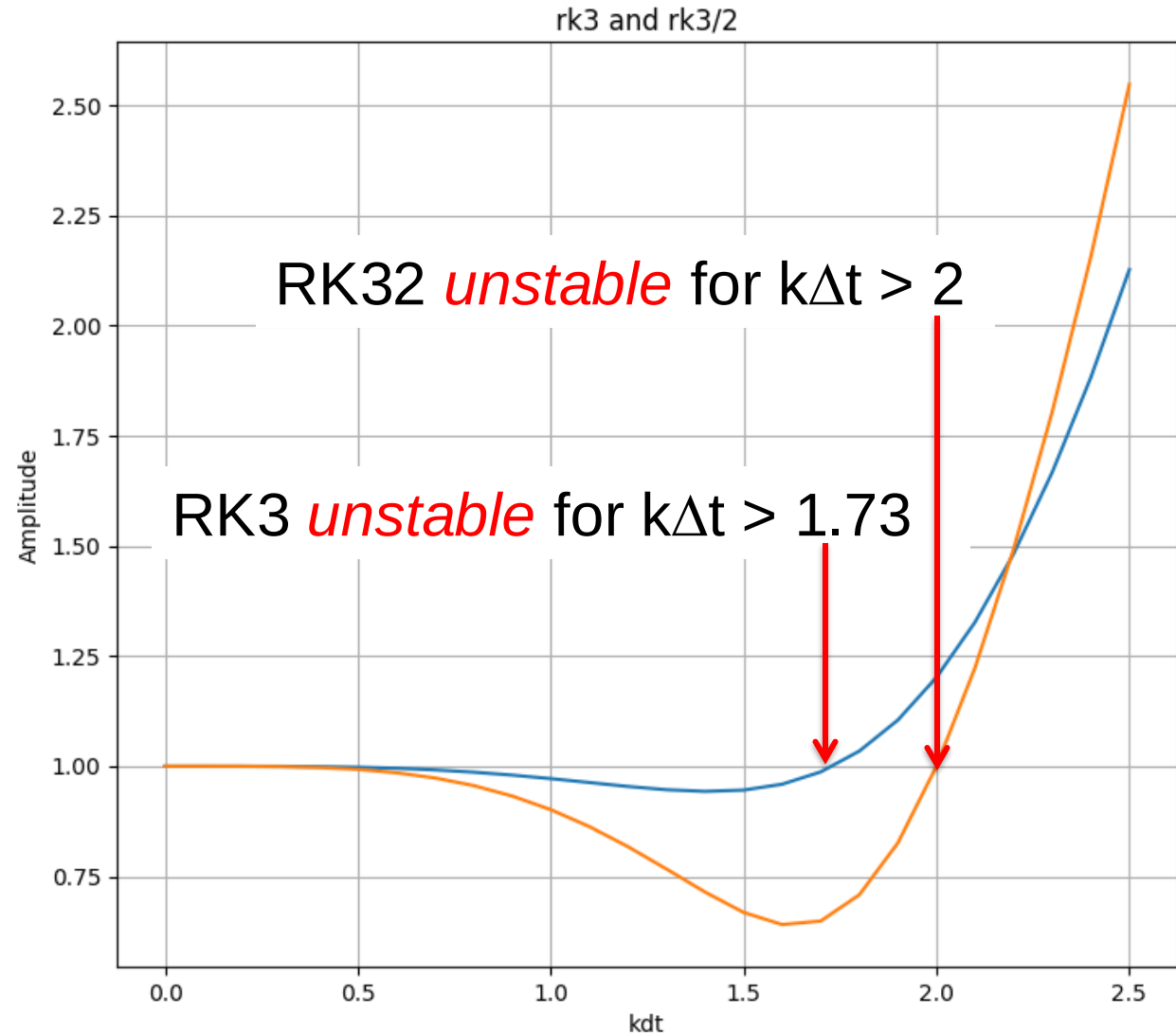
Exact: $|A| = 1$

RK3 and RK32



In applications we see little difference in MPAS solutions using RK3 compared to those using RK32

Time Integration



Time Integration: Acoustic Modes

Split-explicit time integration

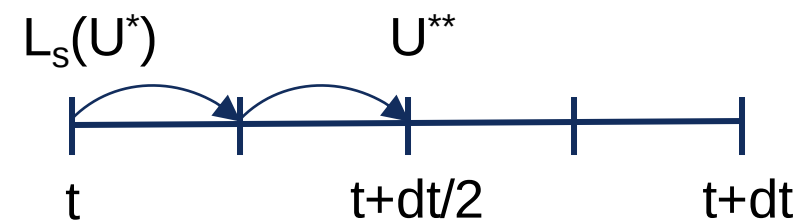
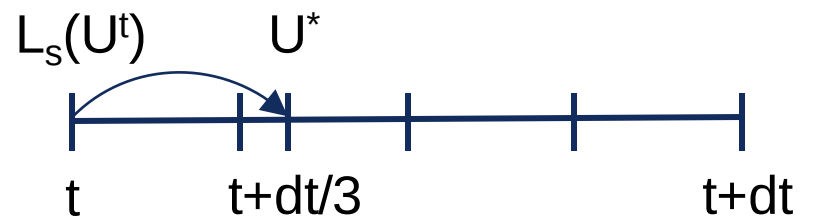
fast: acoustic waves and gravity waves.

slow: everything else.

- RK3 is 3rd order accurate for linear eqns, 2nd order accurate for nonlinear eqns.
- Stable for centered and upwind advection schemes.
- Stable for Courant number $Udt/dx < 1.73$
- Three $L_{\text{slow}}(U)$ evaluations per timestep.

$$U_t = L_{\text{fast}}(U) + L_{\text{slow}}(U)$$

3rd order Runge-Kutta, 3 steps
acoustic steps in RK substeps



Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

Dynamics are integrated first
(`config_split_dynamics_transport = .true.`),
typically with multiple Runge-Kutta
timesteps (`dynamics_split_steps > 1`)

Scalar transport is integrated separately,
after the dynamics

Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

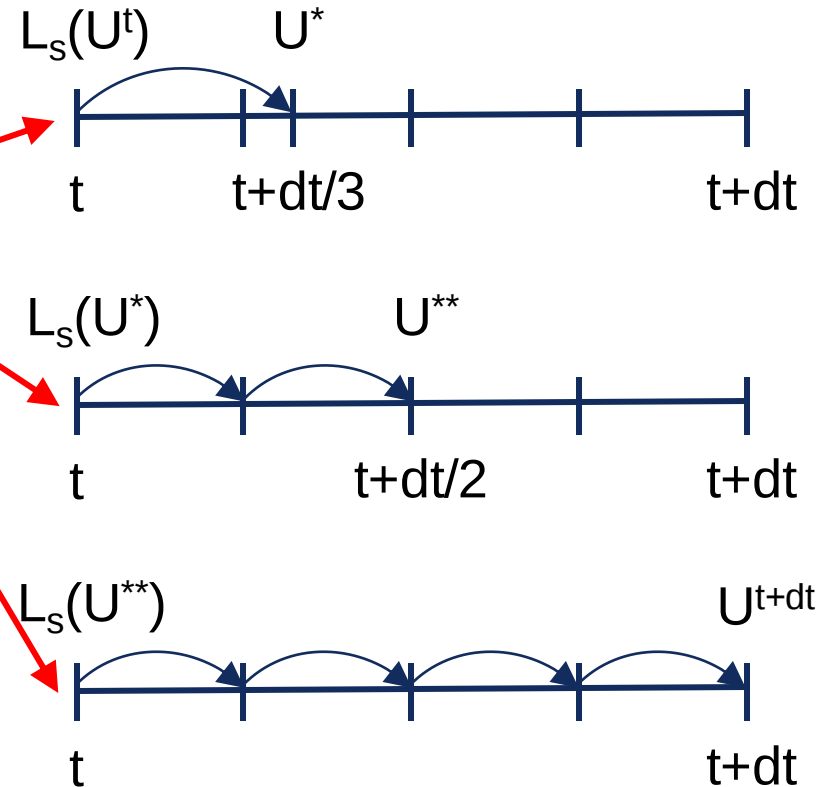
End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

3rd order Runge-Kutta, 3 steps



Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

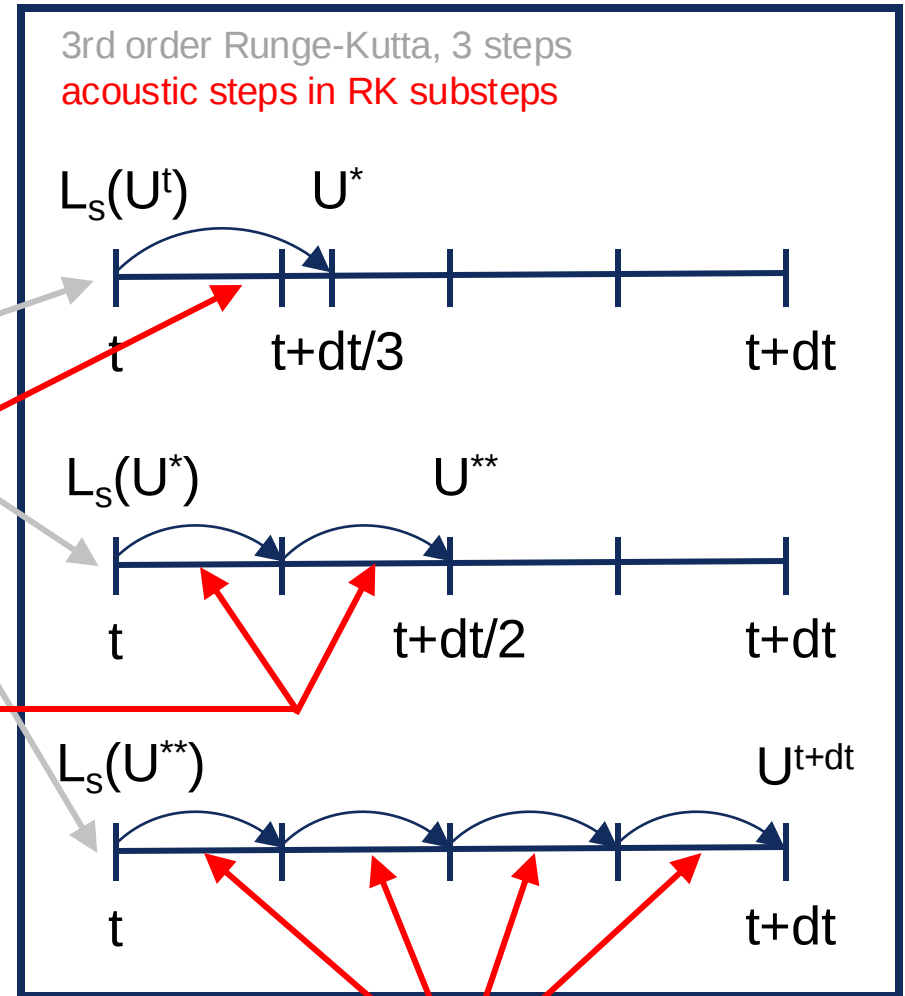
End rk3_step

End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step



Time Integration

Default time integration

3rd order Runge-Kutta, 3 steps
acoustic steps in RK substeps

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

Forward-Backward acoustic mode time integration:

- (1) Explicit integration of the horizontal momentum. There is stability constraint on the acoustic timestep.
- (2) Implicit (in time) integration of the vertically-propagating acoustic modes and gravity waves. There is no stability constraint on the timestep. It uses the result from (1).

Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

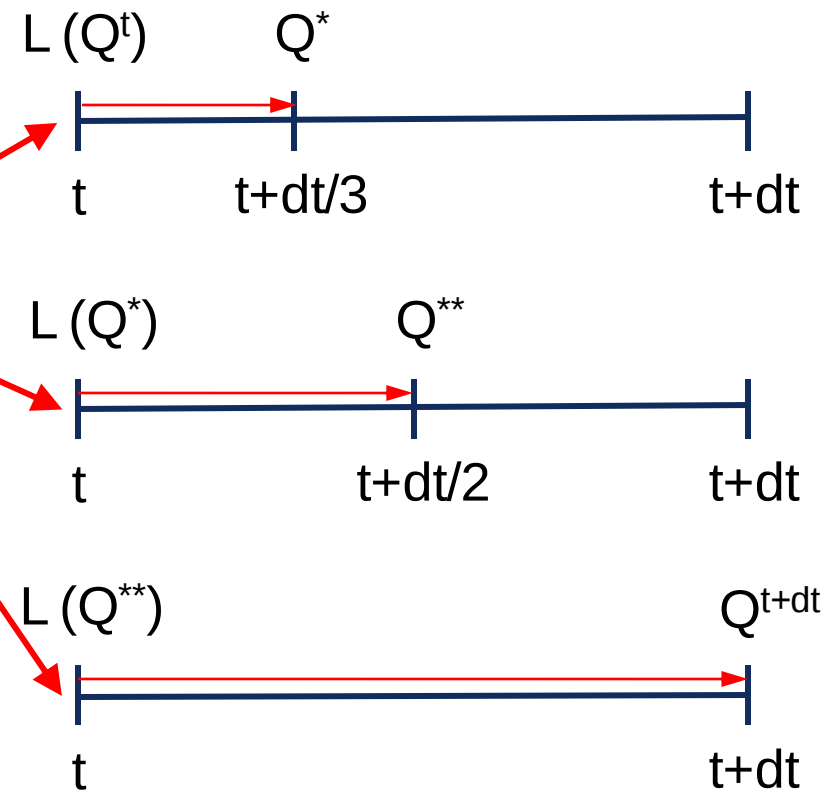
End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

3rd order Runge-Kutta, 3 steps



Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

Allows for smaller dynamics timesteps relative to scalar transport timestep and the main physics timestep.

We can use any transport scheme here (we are not limited to RK3)
Scalar transport and physics are the expensive pieces in most applications.

Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

&nhyd_model

config_dt = 90

config_start_time = "2010-10-23_00:00:00"

config_run_duration = "5_00:00:00"

config_split_dynamics_transport = true

config_dynamics_split_steps = 3

config_number_of_sub_steps = 2

Default time integration

In the file "namelist.atmosphere"

Time Integration

Default time integration

Call physics

Do dynamics_split_steps

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

End rk3_step

End dynamics_split_steps

Do scalar_rk3_step = 1, 3

scalar RK3 transport

End scalar_rk3_step

&nhyd_model

config_dt = 90

config_start_time = "2010-10-23_00:00:00"

config_run_duration = "5_00:00:00"

config_split_dynamics_transport = true

config_dynamics_split_steps = 3

config_number_of_sub_steps = 2

Default time integration

$$\Delta t(\text{dynamics}) = \frac{\text{config_dt}}{\text{config_dynamics_split_steps}}$$

$$\Delta t(\text{acoustic}) = \frac{\Delta t(\text{dynamics})}{\text{config_number_of_sub_steps}}$$

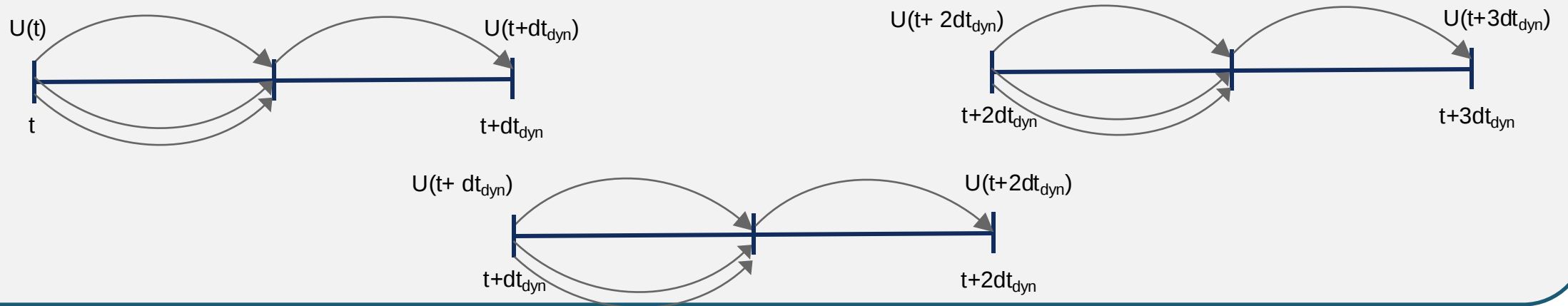
$$\Delta t(\text{scalar transport}) = \text{config_dt}$$

Time Integration

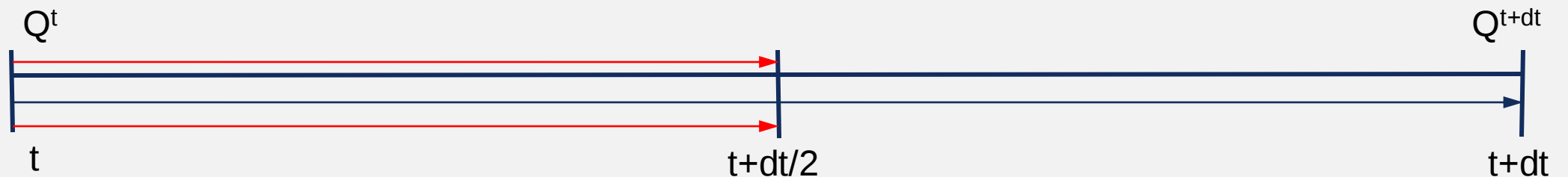
Default configuration summary

config_dynamics_split_steps = 3, config_number_of_sub_steps = 2,
config_time_integration_order = 2

Dynamics timestep



Scalar transport timestep



Time Integration

Option: The WRF approach

```
config_split_dynamics_transport = true/false
config_dynamics_split_steps = 3
config_number_of_sub_steps = 6
                             (acoustic_steps)
```

Time integration option in MPAS

Δt (dynamics) = Δt (scalar transport)
= config_dt

Δt (acoustic) = $\frac{\Delta t \text{ (dynamics)}}{\text{config_number_of_sub_steps}}$

Call physics

Do rk3_step = 1, 3

compute large-time-step tendency

Do acoustic_steps

update u

update rho, theta and w

End acoustic_steps

scalar RK3 transport

End rk3_step

Time Integration

Testing the Timestep Configuration

&nhyd_model

config_dt = 90 ← *Timestep in seconds*

Similar to WRF, the model timestep (in seconds) initially should be set to be 6 times the finest nominal mesh spacing in km. For example – 15 km fine-mesh spacing would use a 90 second timestep.

We have found that a larger timestep is often stable.

If MPAS integrations become unstable (producing NaNs) after just a few timesteps, the issue may be the acoustic modes.

- 1) Reduce the main timestep (*config_dt*) and see if the simulations are stable.
- 2) If stable with a reduced timestep, try the original timestep with a reduced acoustic timestep:
config_number_of_sub_steps > 2 (even integer)
- 3) The acoustic and dry dynamics timestep can also be reduced by increasing *config_dynamics_split_steps* > 3 (can be odd or even)
- 4) If none of these work, then the problem is likely not the dynamics. Check the initial conditions.

Time Integration *References*

Runge-Kutta scheme:

Wicker, L. J., and W. C. Skamarock, 2002: Time Splitting Methods for Elastic Models Using Forward Time Schemes. *Mon. Wea. Rev.*, **130**, 2088-2097.
[https://doi.org/10.1175/1520-0493\(2002\)130<2088:TSMFEM>2.0.CO;2](https://doi.org/10.1175/1520-0493(2002)130<2088:TSMFEM>2.0.CO;2)

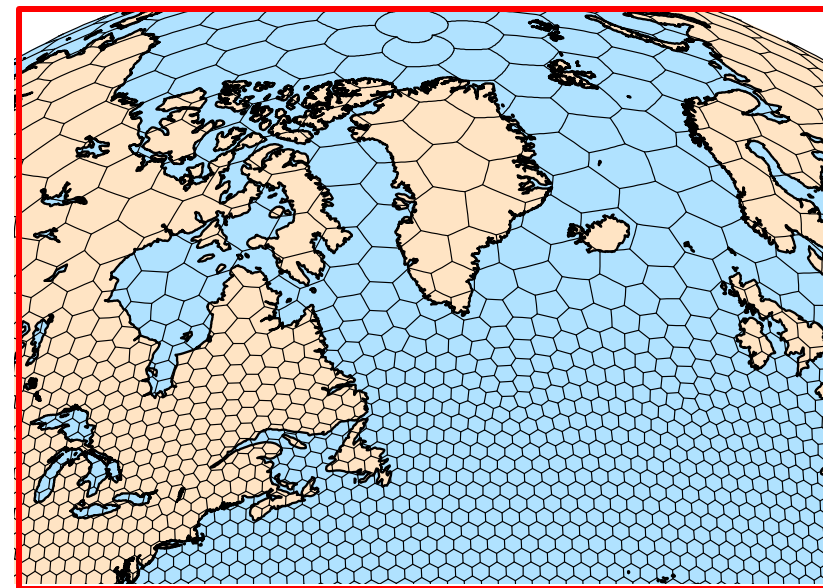
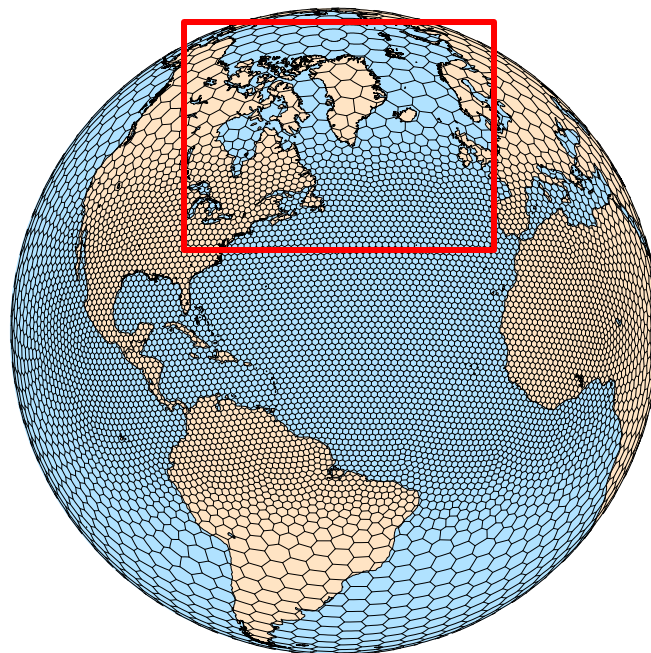
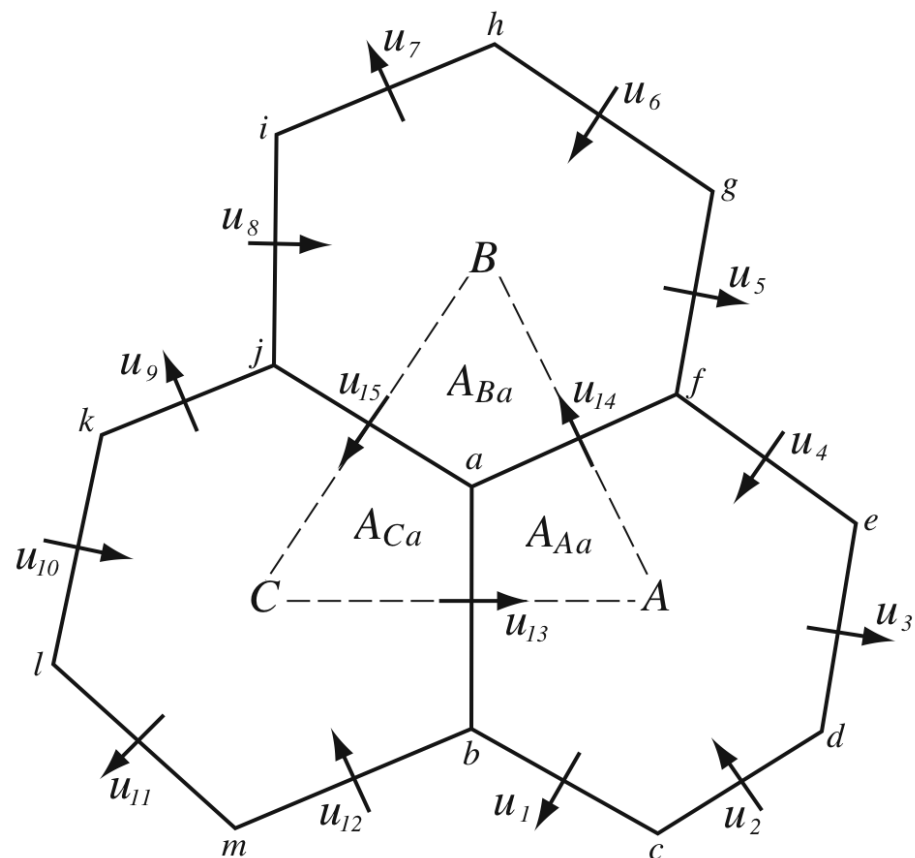
Detailed presentation on the acoustic time splitting:

Klemp, J. B., W. C. Skamarock, and J. Dudhia, 2007: Conservative Split-Explicit Time Integration Methods for the Compressible Nonhydrostatic Equations. *Mon. Wea. Rev.*, **135**, 2897-2913, doi:10.1175/MWR3440.1
(specifically section 2 and Appendix section (a) which deal with height-coordinate models, i.e. MPAS)

MPAS Horizontal Mesh

Unstructured spherical centroidal Voronoi meshes

- Mostly *hexagons*, some pentagons (5-sided cells) and heptagons (7-sided cells).
- Cell centers are at cell center-of-mass (centroidal).
- Cell edges bisect lines connecting cell centers; perpendicular.
- C-grid staggering of velocities (velocities are perpendicular to cell faces).
- Uniform resolution – traditional icosahedral mesh.



MPAS Nonhydrostatic Atmospheric Solver

Equations

- Prognostic equations for coupled variables.
- Generalized height coordinate.
- Horizontally vector-invariant equation set.
- Continuity equation for dry air mass.
- Thermodynamic equation for coupled potential temperature.

Variables: $(U, V, \Omega, \Theta, Q_j) = \tilde{\rho}_d(u, v, \omega, \theta, q_j)$ $\tilde{\rho}_d = \rho_d/\zeta_z$

Vertical coordinate: $z = \zeta + A(\zeta)h_s(x, y, \zeta)$

Prognostic equations:

$$\frac{\partial \mathbf{V}_H}{\partial t} = - \frac{\rho_d}{\rho_m} \left[\nabla_\zeta \left(\frac{p}{\zeta_z} \right) - \frac{\partial \mathbf{z}_H p}{\partial \zeta} \right] - \eta \mathbf{k} \times \mathbf{V}_H - \mathbf{v}_H \nabla_\zeta \cdot \mathbf{V} - \frac{\partial \Omega \mathbf{v}_H}{\partial \zeta} - \rho_d \nabla_\zeta K + \mathbf{F}_{V_H}$$

$$\frac{\partial W}{\partial t} = - \frac{\rho_d}{\rho_m} \left[\frac{\partial p}{\partial \zeta} + g \tilde{\rho}_m \right] - (\nabla \cdot \mathbf{v} W)_\zeta + F_W$$

$$\frac{\partial \Theta_m}{\partial t} = - (\nabla \cdot \mathbf{V} \theta_m)_\zeta + F_{\Theta_m}$$

$$\frac{\partial \tilde{\rho}_d}{\partial t} = - (\nabla \cdot \mathbf{V})_\zeta$$

$$\frac{\partial Q_j}{\partial t} = - (\nabla \cdot \mathbf{V} q_j)_\zeta + F_{Q_j}$$

$$\frac{\rho_m}{\rho_d} = 1 + q_v + q_c + q_r + \dots$$

Diagnostics and definitions:

$$p = p_0 \left(\frac{R_d \zeta_z \Theta_m}{p_0} \right)^\gamma \quad \theta_m = \theta [1 + (R_v/R_d)q_v]$$

Gradient operators

Nonlinear Coriolis term

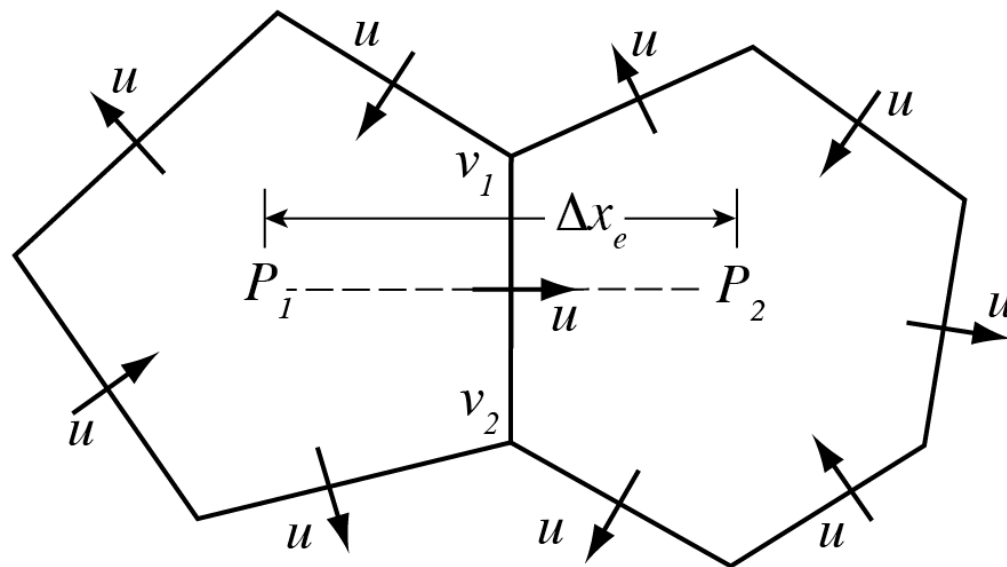
Operators on the Voronoi Mesh

Pressure and KE gradients

$$\frac{\partial \mathbf{V}_H}{\partial t} = - \frac{\rho_d}{\rho_m} \left[\nabla_\zeta \left(\frac{p}{\zeta_z} \right) - \frac{\partial \mathbf{z}_H p}{\partial \zeta} \right] - \eta \mathbf{k} \times \mathbf{V}_H$$

$$- \mathbf{v}_H \nabla_\zeta \cdot \mathbf{V} - \frac{\partial \Omega \mathbf{v}_H}{\partial \zeta} - \rho_d \nabla_\zeta K + \mathbf{F}_{V_H}$$

On the Voronoi mesh, $P_1 P_2$ is perpendicular to $v_1 v_2$ and is bisected by $v_1 v_2$, hence $P_x \sim (P_2 - P_1) \Delta x_e^{-1}$ is 2nd order accurate.



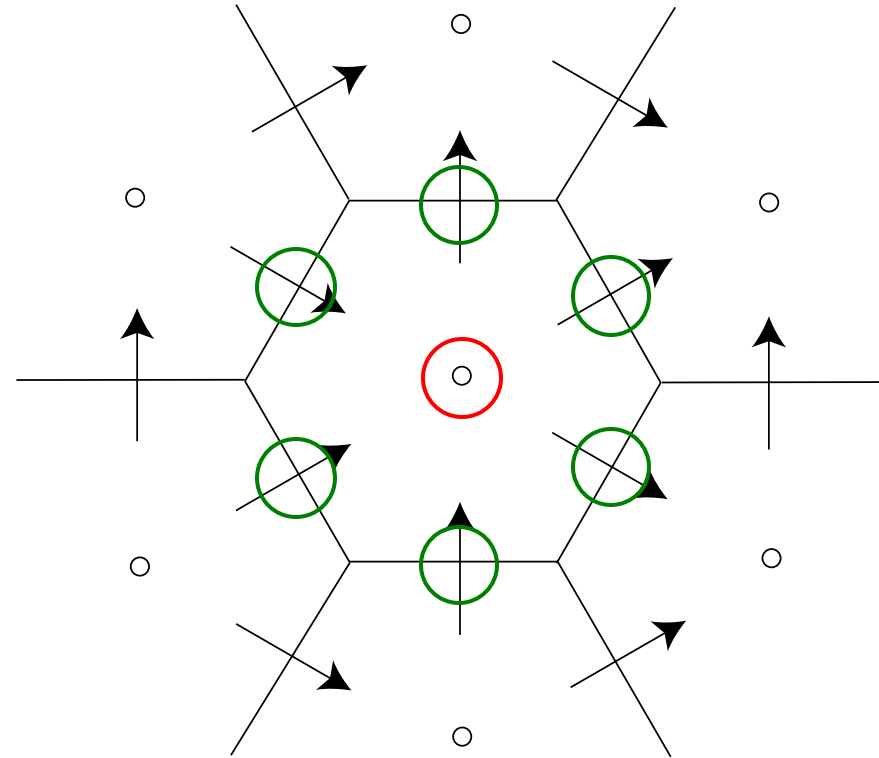
Operators on the Voronoi Mesh *Pressure and KE gradients*

Cell center kinetic energy: KE_i

$$KE_i = (1 - b) \sum_{e_i} w_{e_i} u_{e_i}^2 + b \sum_{v_j} w_{v_j} KE_{v_j}$$

Vertex kinetic energy: KE_v

$$KE_v = \sum_{e_v=1}^3 w_{e_v} u_{e_v}^2$$



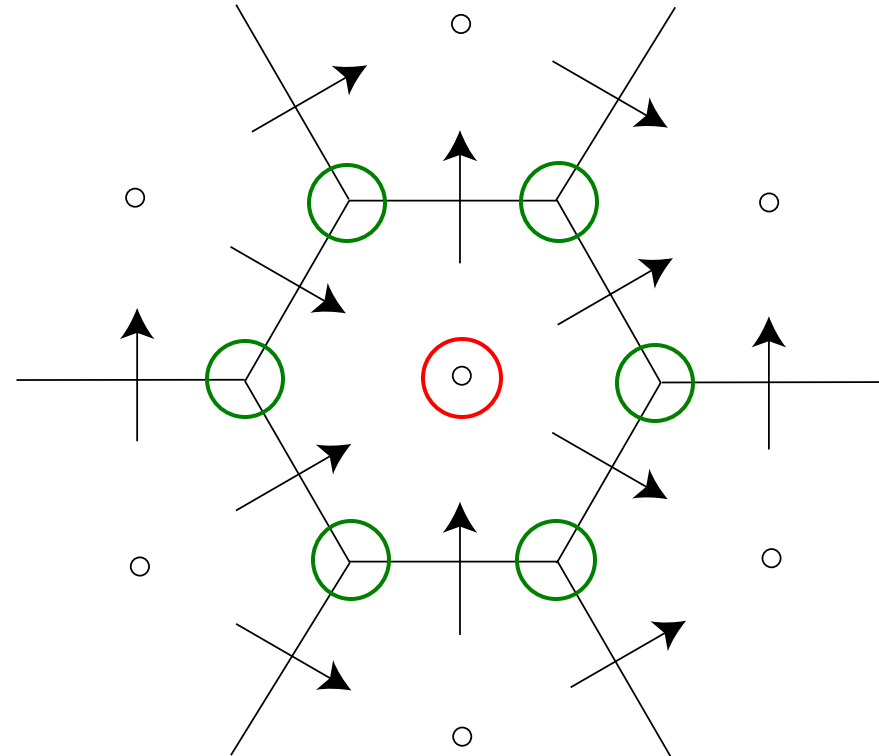
Operators on the Voronoi Mesh *Pressure and KE gradients*

Cell center kinetic energy: KE_i

$$KE_i = (1 - b) \sum_{e_i} w_{e_i} u_{e_i}^2 + b \sum_{v_j} w_{v_j} KE_{v_j}$$

Vertex kinetic energy: KE_v

$$KE_v = \sum_{e_v=1}^3 w_{e_v} u_{e_v}^2$$



Operators on the Voronoi Mesh

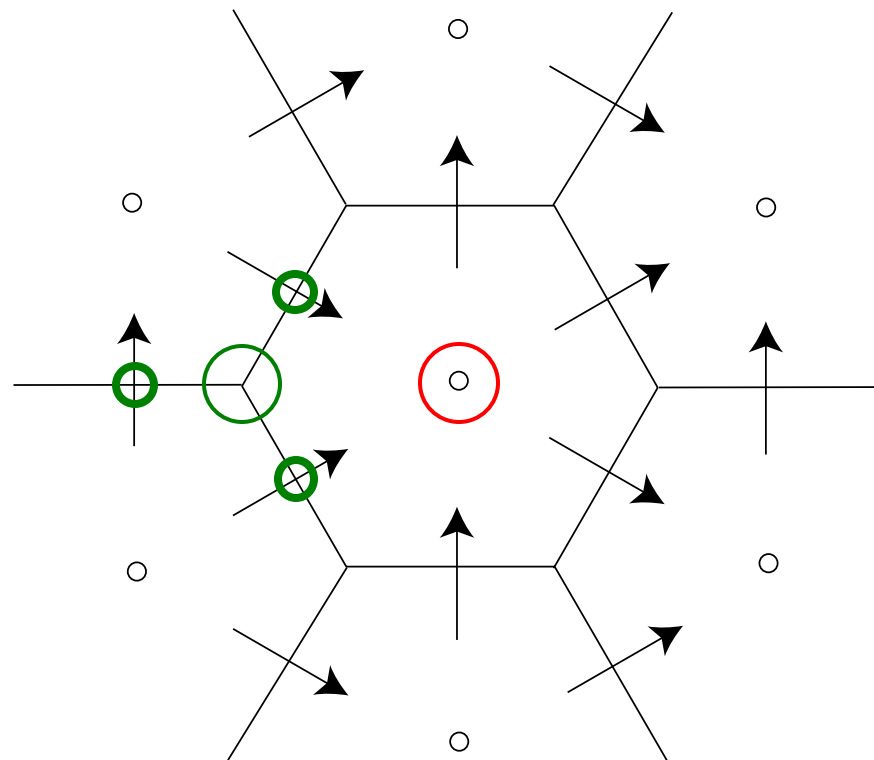
Pressure and KE gradients

Cell center kinetic energy: KE_i

$$KE_i = (1 - b) \sum_{e_i} w_{e_i} u_{e_i}^2 + b \sum_{v_j} w_{v_j} KE_{v_j}$$

Vertex kinetic energy: KE_v

$$KE_v = \sum_{e_v=1}^3 w_{e_v} u_{e_v}^2$$



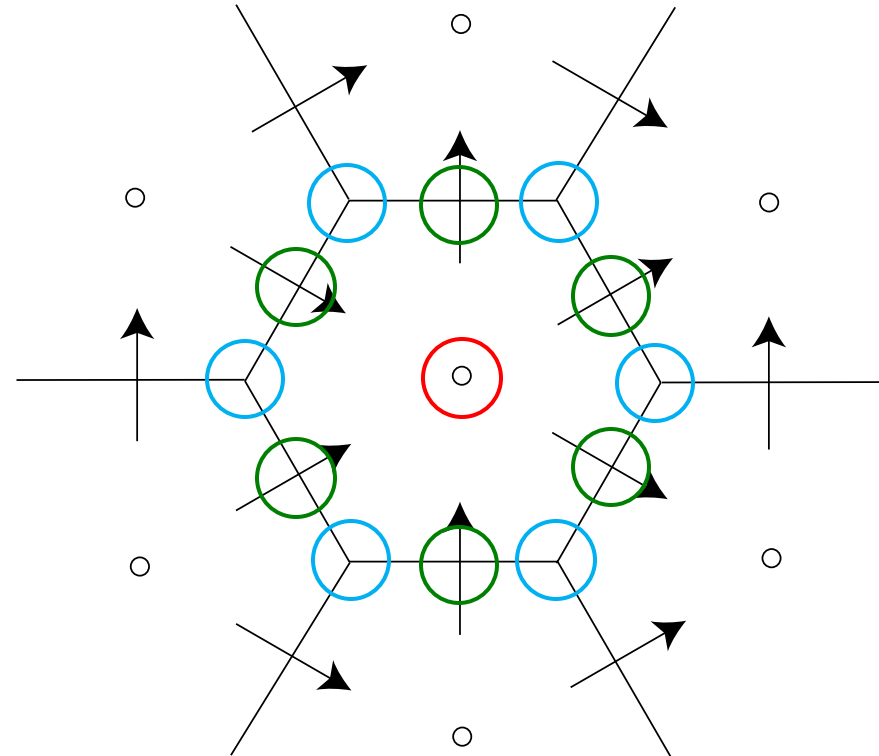
Operators on the Voronoi Mesh *Pressure and KE gradients*

Cell center kinetic energy: KE_i

$$KE_i = (1 - b) \sum_{e_i} w_{e_i} u_{e_i}^2 + b \sum_{v_j} w_{v_j} KE_{v_j}$$

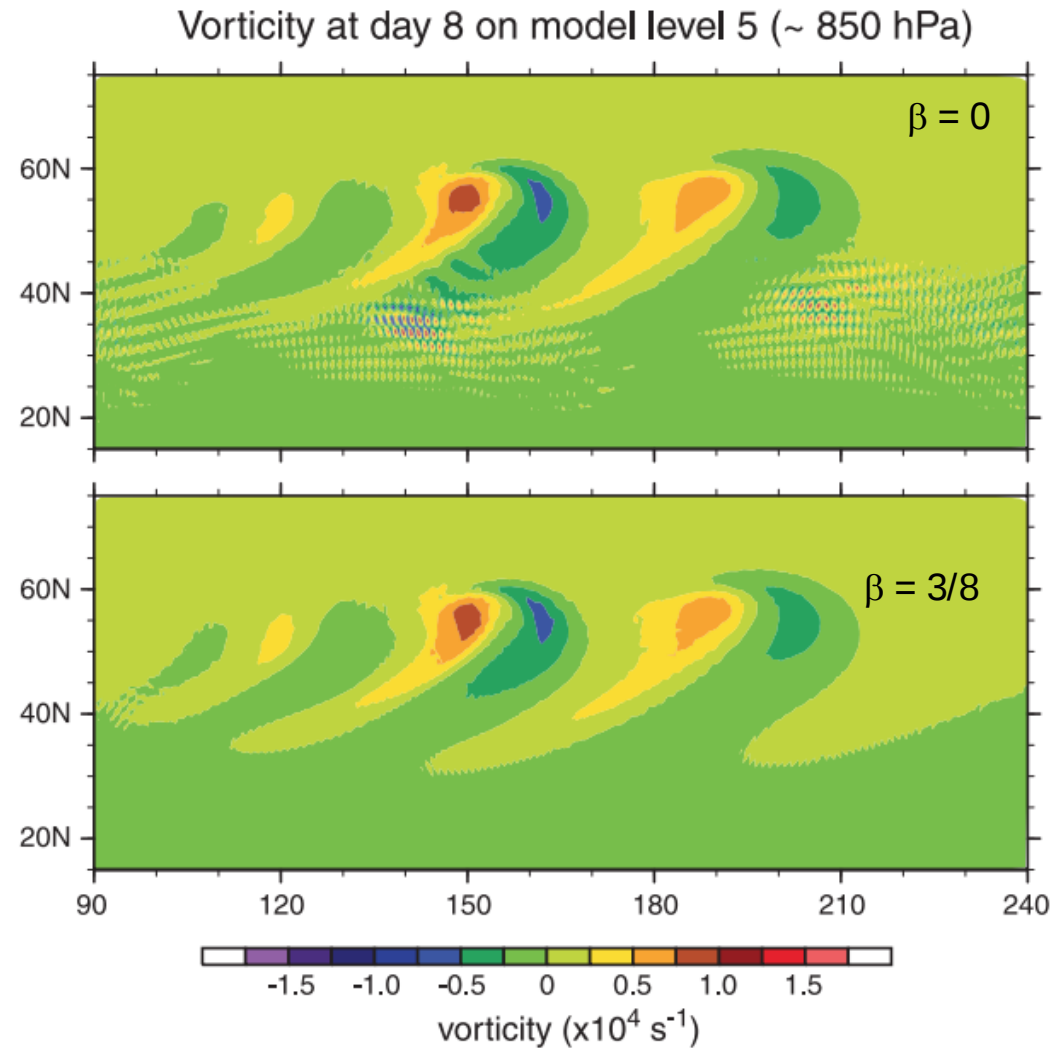
Vertex kinetic energy: KE_v

$$KE_v = \sum_{e_v=1}^3 w_{e_v} u_{e_v}^2$$



Operators on the Voronoi Mesh *cell-center KE evaluation*

MPAS uses $\beta = 3/8$



Operators on the Voronoi Mesh 'Nonlinear' Coriolis force

Tangential
velocity
reconstruction:

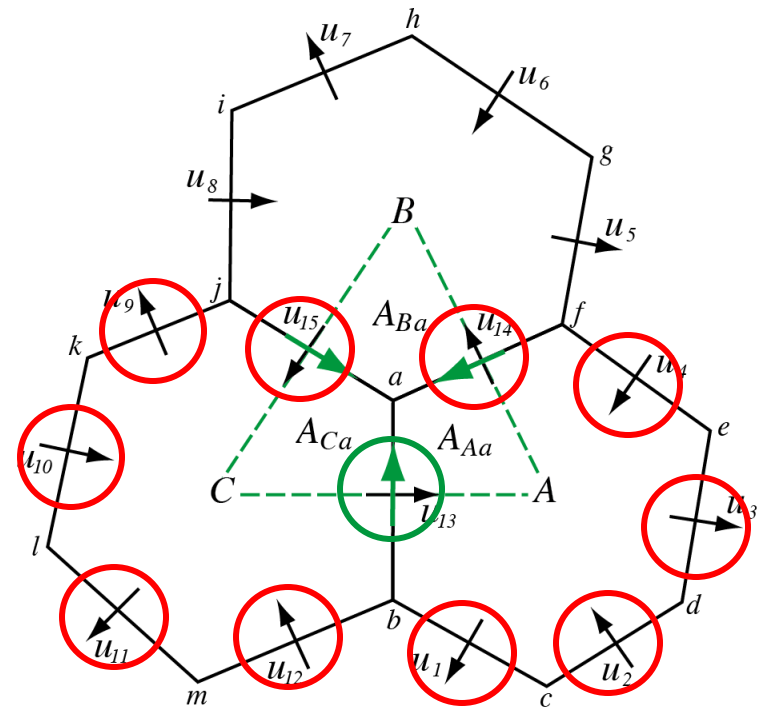
$$\mathbf{v}_{e_i} = \sum_{j=1}^{n_{e_i}} w_{e_i,j} \mathbf{u}_{e_i,j}$$

$$\frac{\partial \mathbf{V}_H}{\partial t} = - \frac{\rho_d}{\rho_m} \left[\nabla_{\zeta} \left(\frac{p}{\zeta_z} \right) - \frac{\partial \mathbf{z}_{HP}}{\partial \zeta} \right] - \eta \mathbf{k} \times \mathbf{V}_H \\ - \mathbf{v}_H \nabla_{\zeta} \cdot \mathbf{V} - \frac{\partial \Omega \mathbf{v}_H}{\partial \zeta} - \rho_d \nabla_{\zeta} K + \mathbf{F}_{V_H}$$

Nonlinear term:

$$[\eta \mathbf{k} \times \mathbf{V}_H]_{e_i} = \sum_{j=1}^{n_{e_i}} \frac{1}{2} (\eta_{e_i} + \eta_{e_i,j}) w_{e_i,j} \rho_{e_i,j} \mathbf{u}_{e_i,j}$$

The general tangential velocity reconstruction produces a consistent divergence on the primal and dual grids, and allows for PV, enstrophy and energy* conservation in the nonlinear SW solver.



Operators on the Voronoi Mesh *'Nonlinear' Coriolis force*

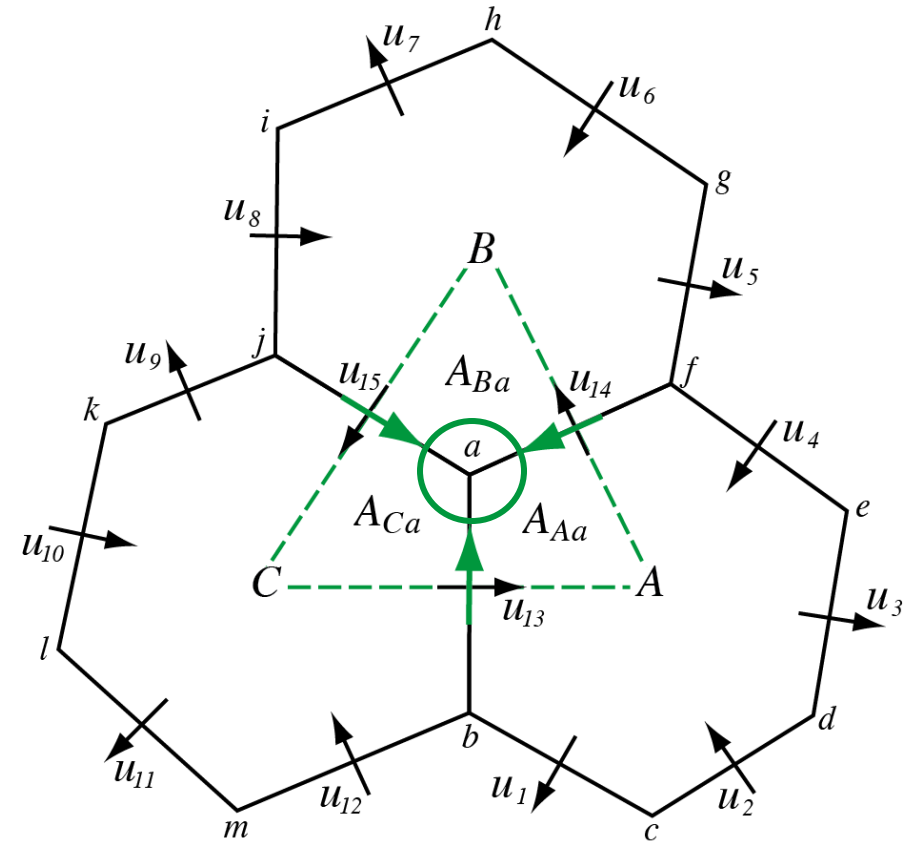
$$[\eta \mathbf{k} \times \mathbf{V}_H]_{e_i} = \sum_{j=1}^{n_{e_i}} \frac{1}{2} (\eta_{e_i} + \eta_{e_{i,j}}) w_{e_{i,j}} \rho_{e_{i,j}} \mathbf{u}_{e_{i,j}}$$

Example: absolute vorticity at e_{13}

$$h_{13} = \frac{1}{2} (h_a + h_b)$$

Example: absolute vorticity at vertex a

$$\eta_a = f_a + \frac{(u_{13} |\overline{CA}| + u_{14} |\overline{AB}| + u_{15} |\overline{BC}|)}{\text{Area}(ABC)}$$



Configuring the dynamics

(*namelist.atmosphere*)

&nhyd_model

config_apvm_upwinding = 0.5

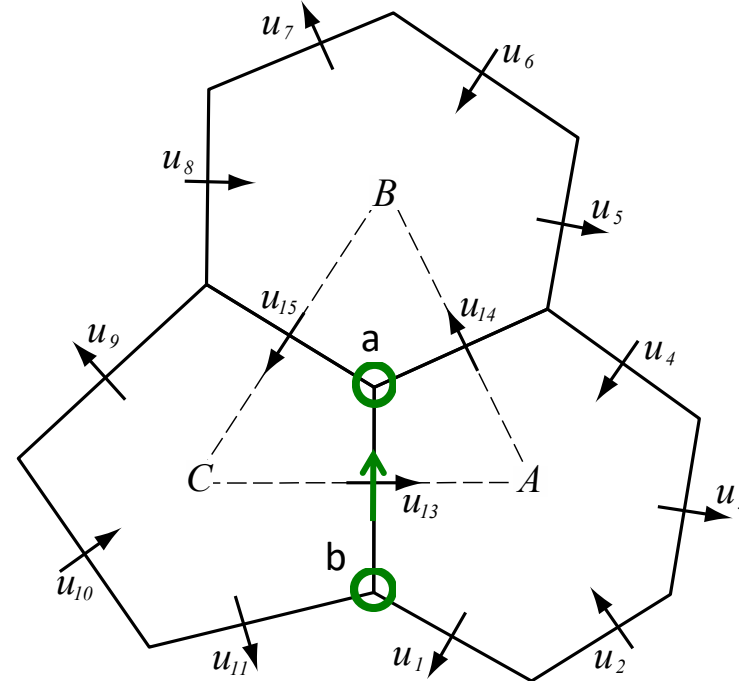
Upwind estimate of vorticity at the cell faces using a timestep of *config_apvm_upwinding* Δt

$$\frac{\partial \mathbf{V}_H}{\partial t} = -\frac{\rho_d}{\rho_m} \left[\nabla_\zeta \left(\frac{p}{\zeta_z} \right) - \frac{\partial \mathbf{z}_{HP}}{\partial \zeta} \right] - \eta \mathbf{k} \times \mathbf{V}_H$$

$$- \mathbf{v}_H \nabla_\zeta \cdot \mathbf{V} - \frac{\partial \Omega \mathbf{v}_H}{\partial \zeta} - \rho_d \nabla_\zeta K + \mathbf{F}_{VH}$$

$$[\eta \mathbf{k} \times \mathbf{V}_H]_{e_i} = \sum_{j=1}^{n_{e_i}} \frac{1}{2} (\eta_{e_i} + \eta_{e_{i,j}}) w_{e_{i,j}} \rho_{e_{i,j}} u_{e_{i,j}}$$

Vorticity at cell faces (at u points)



APVM: Anticipated Potential Vorticity Method (Sadourny 1985)

Upwinding the vorticity here will result in dissipation of the vorticity.

Vorticity at edge 13:

$$\text{config_apvm_upwinding} = 0, \eta_{13} = (\eta_a + \eta_b)$$

$$\text{config_apvm_upwinding} = 0.5, \eta_{13} = (\eta_a + \eta_b) - 0.5 \Delta t (u_e \eta_x - v_e \eta_y)$$

Spatial Discretization in MPAS

references

Dynamics

Skamarock, W. C., J. B. Klemp, M. G. Duda, L. Fowler, S.-H. Park, and T. D. Ringler, 2012: A Multi-scale Nonhydrostatic Atmospheric Model Using Centroidal Voronoi Tessellations and C-Grid Staggering. *Mon. Wea. Rev.*, 140, 3090-3105. doi:10.1175/MWR-D-11-00215.1

Ringler, T. D., J. Thuburn, J.B. Klemp, W. C. Skamarock, 2010: A unified approach to energy conservation and potential vorticity dynamics for arbitrarily-structured C-grids. *J. Comp. Phys.*, 229, 3065-3090. doi:10.1016/j.jcp.2009.12.007

The Most Important Takeaway from this Lecture

Testing the Timestep Configuration

&nhyd_model

config_dt = 90 ← *Timestep in seconds*

Similar to WRF, the model timestep (in seconds) initially should be set to be 6 times the finest nominal mesh spacing in km. For example – 15 km fine-mesh spacing would use a 90 second timestep.

We have found that *a larger timestep is often stable*.

If MPAS integrations become unstable (producing NaNs) after just a few timesteps, the issue may be the acoustic modes.

- 1) Reduce the main timestep (*config_dt*) and see if the simulations are stable.
- 2) If stable with a reduced timestep, try the original timestep with a reduced acoustic timestep:
config_number_of_sub_steps > 2 (even integer)
- 3) The acoustic and dry dynamics timestep can also be reduced by increasing *config_dynamics_split_steps* > 3 (can be odd or even)
- 4) If none of these work, then the problem is likely not the dynamics. Check the initial conditions.