The Advanced Research WRF (ARW) Dynamics Solver

1. What is a dynamics solver/dynamical core?
2. Variables and coordinates
3. Equations
4. Time integration scheme
5. Grid staggering
6. Advection (transport) and conservation
7. Time step parameters
8. Filters and filter parameters
9. Map projections and global configuration
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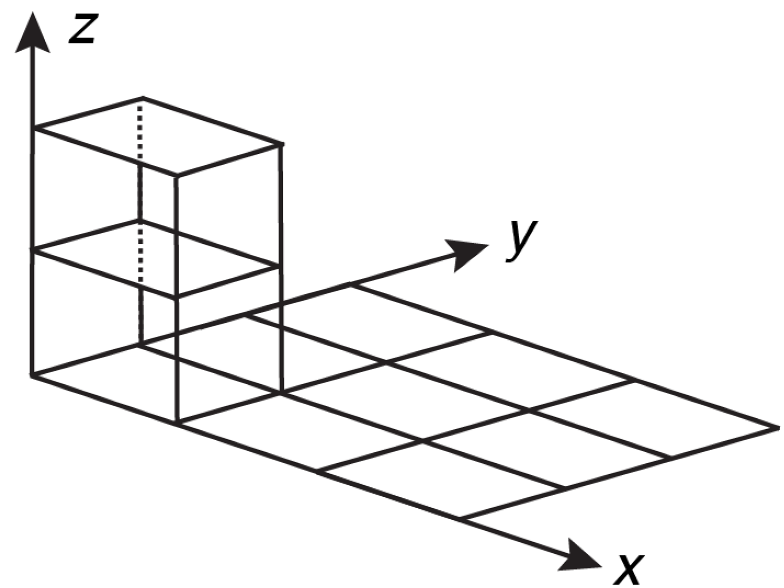
WRF ARW Tech Note
A Description of the Advanced Research WRF Version 4 (March 2021; WRF Version 4.3)
http://www2.mmm.ucar.edu/wrf/users/docs/technote/contents.html

A *dynamical solver* (or a *dynamical core*, or *dycore*) performs a time (t) and space (x,y,z) integration of the equations of motion.

Given the 3D atmospheric state at time t, $S(x,y,z,t)$, we integrate the equations forward in time from $t \longrightarrow T$, i.e. we run the model and produce a forecast.

The equations cannot be solved analytically, so we *discretize* the equations on a *grid* and compute *approximate* solutions.



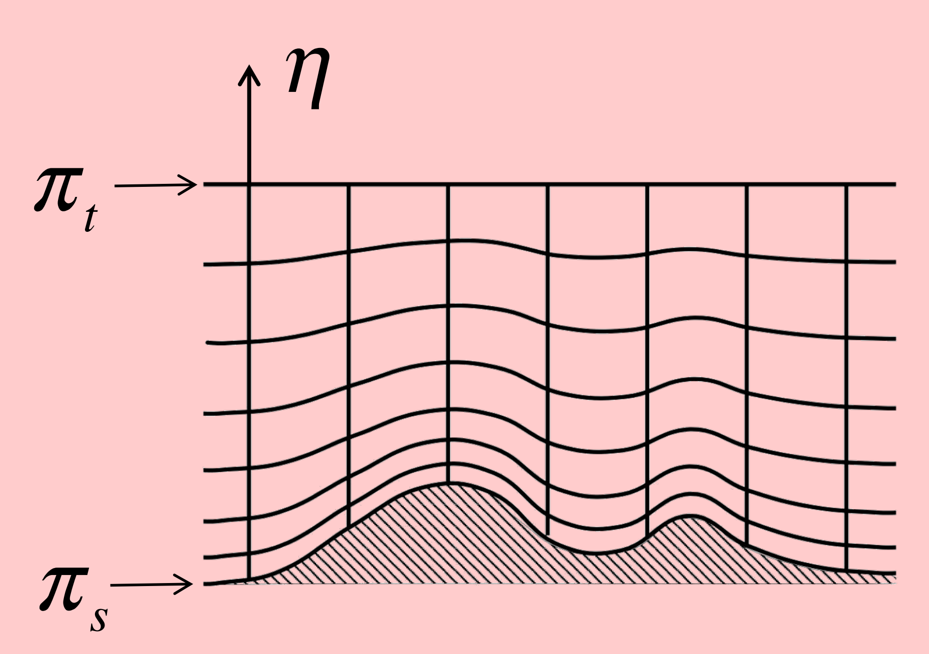
The accuracy of the solutions depend on the numerical method and the mesh spacing (grid).

Vertical coordinates: (1) Traditional terrain-following mass coordinate

Dry hydrostatic pressure π_d

Column mass (per unit area) $\mu_d = \pi_s - \pi_t$

Vertical coordinate $\eta = \dfrac{(\pi_d - \pi_t)}{\mu_d}$



Layer mass (per unit area) $\mu_d \Delta\eta = \Delta\pi_d = -g\rho_d \Delta z$, Pressure $\pi_d(\eta) = \eta\mu_d + \pi_t$

Vertical coordinates: (2) Hybrid terrain-following mass coordinate

Isobaric coordinate (constant pressure):

$$\eta = \frac{\pi_d}{\pi_0 - \pi_t}$$

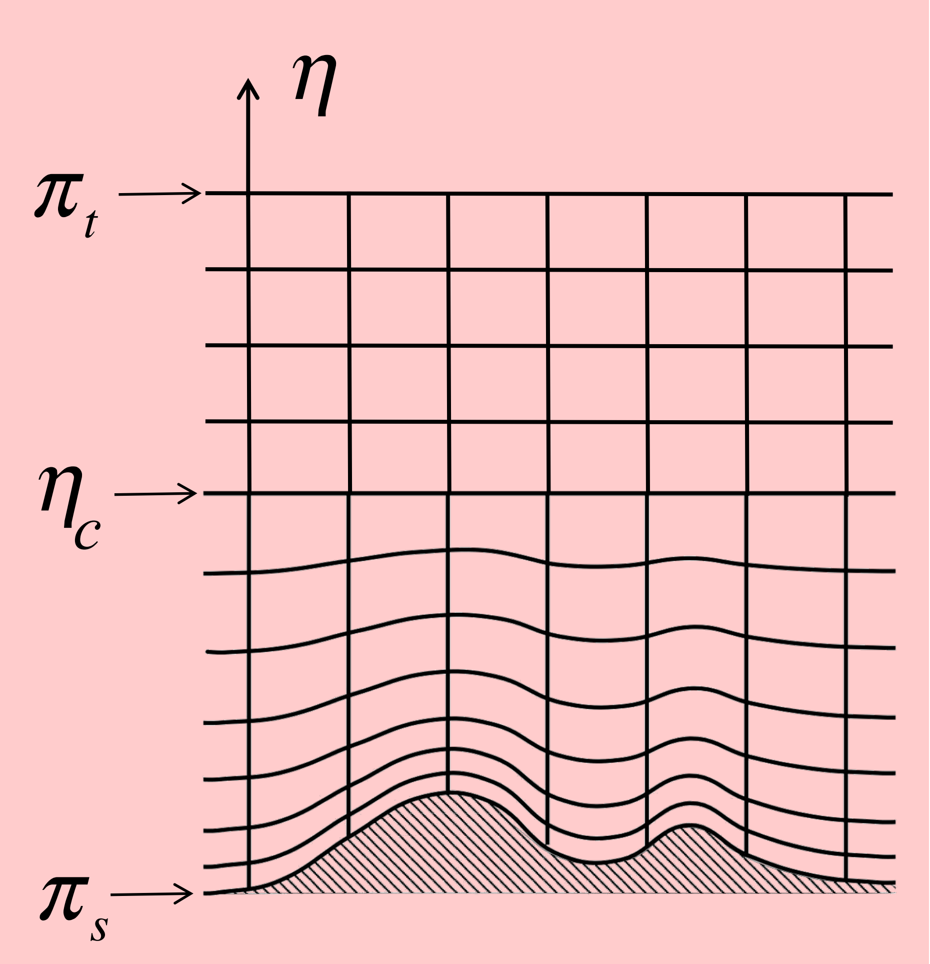
Hybrid terrain-following coordinate:

$$\pi_d(\eta) = B(\eta)\mu_d + \pi_t \quad \text{(Terrain-following)}$$

$$+[\eta - B(\eta)](\pi_0 - \pi_t) \quad \text{(Isobaric)}$$

Level at which $B(\eta) \rightarrow 0$ is the transition between isobaric and terrain-following coordinate.

Default WRFV4 configuration:
Hybrid coordinate is enabled, $\eta_c = 0.2$



Variables:

Grid volume mass (per unit area):

$$\mu_d = \frac{\partial \pi_d}{\partial \eta} = B_\eta(\pi_s - \pi_t) + (1 - B_\eta)(\pi_0 - \pi_t)$$

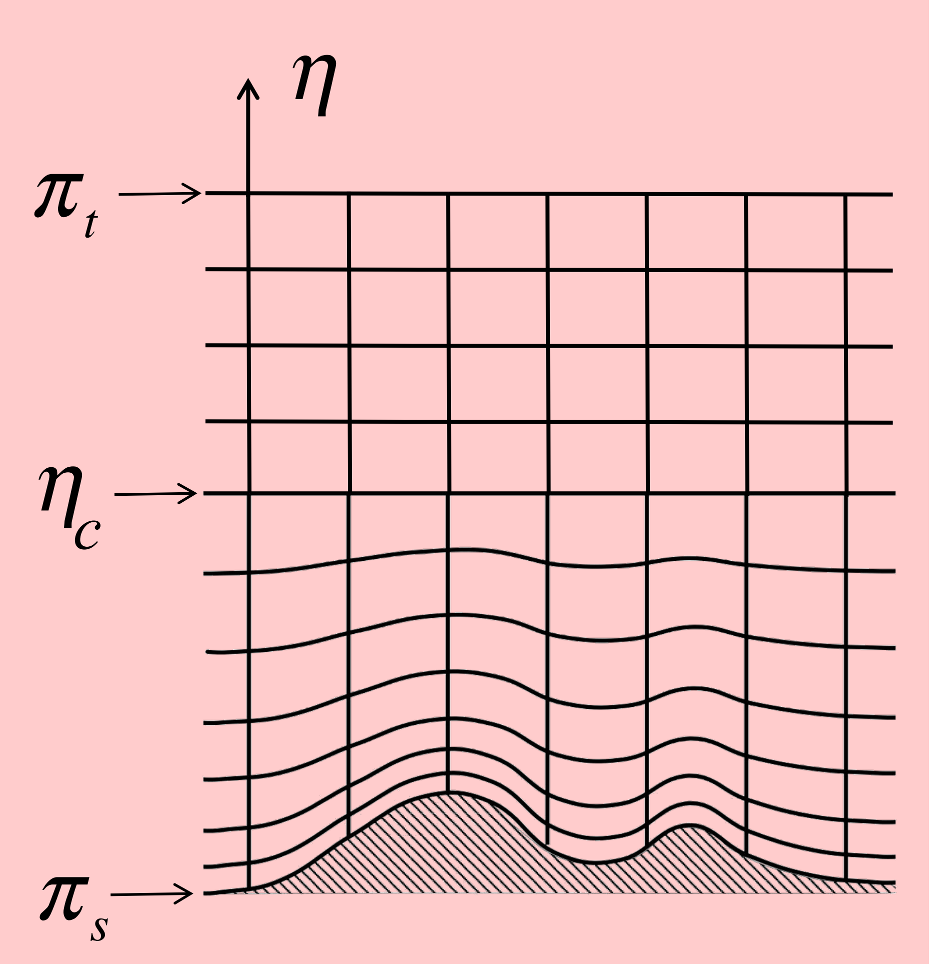
Conserved state (prognostic) variables:

$$\mu_d, \quad U = \mu_d u, \quad V = \mu_d v,$$

$$W = \mu_d w, \quad \Theta = \mu_d \theta$$

Non-conserved state variable:

$$\phi = gz$$



Dynamics: 2. Variables and coordinates

Diagnostic relations: $$\frac{\partial\phi}{\partial\eta} = -\alpha_d \mu_d, \quad p = \left(\frac{R_d \Theta_m}{p_o \mu_d \alpha_d}\right)^{\gamma}, \quad \Theta_m = \Theta\left(1 + \frac{R_v}{R_d} q_v\right)$$

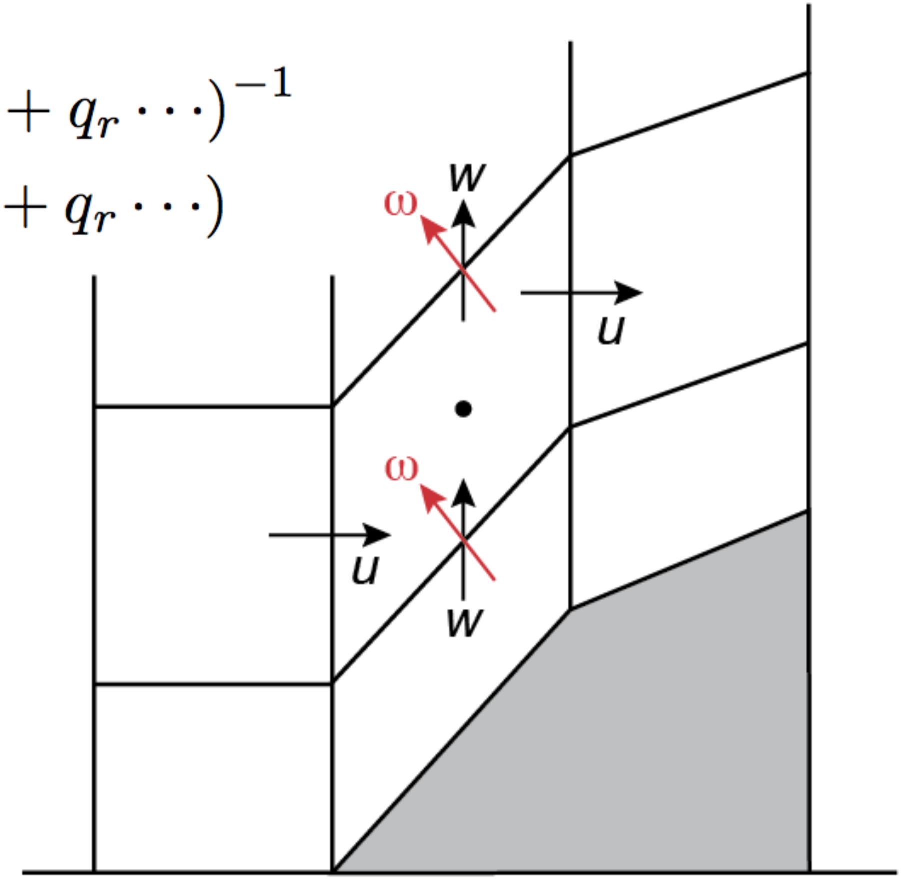
Subscript *d* denotes *dry*, and

$$\alpha_d = \frac{1}{\rho_d} \qquad \begin{aligned} \alpha &= \alpha_d \left(1 + q_v + q_c + q_r \cdots\right)^{-1} \\ \rho &= \rho_d \left(1 + q_v + q_c + q_r \cdots\right) \end{aligned}$$

covariant (u, ω) and
contravariant w velocities

$$u = \frac{dx}{dt}, \quad w = \frac{dz}{dt}, \quad \omega = \frac{d\eta}{dt}$$

$$U = \mu u, \quad W = \mu w, \quad \Omega = \mu\omega$$



Dynamics: 3. Equations

$$\frac{\partial U}{\partial t} =$$

$$\frac{\partial V}{\partial t} =$$

$$\frac{\partial W}{\partial t} =$$

$$\frac{\partial \mu_d}{\partial t} =$$

$$\frac{\partial \Theta}{\partial t} =$$

$$\frac{\partial \mu_d q_j}{\partial t} =$$

$$\frac{\partial \phi}{\partial t} =$$

transport

$$\frac{\partial U}{\partial t} = -\frac{\partial U u}{\partial x} - \frac{\partial V u}{\partial y} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial V}{\partial t} = -\frac{\partial U v}{\partial x} - \frac{\partial V v}{\partial y} - \frac{\partial \Omega v}{\partial \eta}$$

$$\frac{\partial W}{\partial t} = -\frac{\partial U w}{\partial x} - \frac{\partial V w}{\partial y} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U \theta}{\partial x} - \frac{\partial V \theta}{\partial y} - \frac{\partial \Omega \theta}{\partial \eta}$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta}$$

transport pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y}$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right)$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta}$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial Uq_j}{\partial x} - \frac{\partial Vq_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta}$$

Dynamics: 3. Equations

transport — pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta} + R_\theta + Q_\theta$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

R, Q terms: numerical filters, physics, projection terms

$+gw$: geopotential eqn term

Dynamics: 3. Equations

transport — pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta} + R_\theta + Q_\theta$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

numerical filters, physics, projection terms (R, Q terms)

geopotential eqn term ($+gw$)

Diagnostic relations: $\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d$, $p = \left(\frac{R_d \Theta_m}{p_o \mu_d \alpha_d}\right)^\gamma$, $\Theta_m = \Theta\left(1 + \frac{R_v}{R_d} q_v\right)$

3rd Order Runge-Kutta time integration

$$\frac{\partial U}{\partial t} = RHS_u$$

$$\frac{\partial V}{\partial t} = RHS_v$$

$$\frac{\partial W}{\partial t} = RHS_w$$

•
•
•

advance $\phi^t \rightarrow \phi^{t+\Delta t}$

$$\phi^* = \phi^t + \frac{\Delta t}{3} RHS(\phi^t)$$

$$\phi^{**} = \phi^t + \frac{\Delta t}{2} RHS(\phi^*)$$

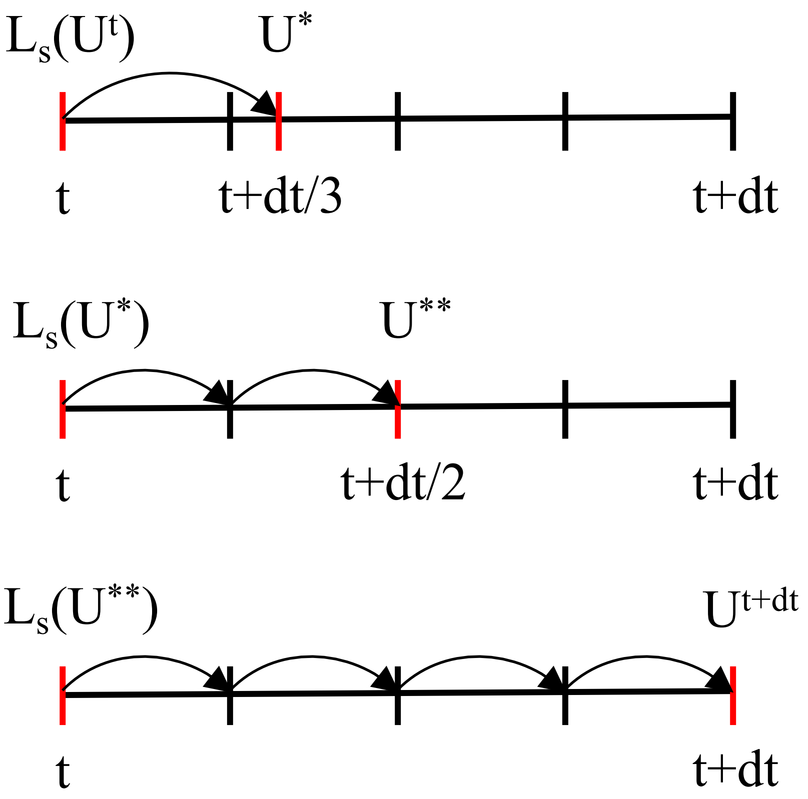
$$\phi^{t+\Delta t} = \phi^t + \Delta t \, RHS(\phi^{**})$$

Amplification factor $\phi_t = ik\phi; \quad \phi^{n+1} = A\phi^n; \; |A| = 1 - \frac{(k\Delta t)^4}{24}$

$$U_t = L_{fast}(U) + L_{slow}(U)$$

fast: acoustic and gravity wave terms.
slow: everything else.

3rd order Runge-Kutta, 3 steps



- RK3 is 3rd order accurate for linear eqns, 2nd order accurate for nonlinear eqns.
- Stable for centered and upwind advection schemes.
- Stable for Courant number $Udt/dx < 1.73$
- Three $L_{slow}(U)$ evaluations per timestep.

Dynamics: 4. Time integration scheme – acoustic step

$$U^{t+\Delta t}, \quad V^{t+\Delta t} \qquad \frac{\partial U}{\partial t} + \left(\mu_d \alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \phi}{\partial x} \right)^{\tau} = R_U^t$$

$$\mu_d^{\tau+\Delta\tau} \quad \Omega^{\tau+\Delta\tau} \qquad \frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x}^{\tau+\Delta\tau} + \frac{\partial \Omega}{\partial \eta}^{\tau+\Delta\tau} = 0$$

$$\Theta^{\tau+\Delta\tau} \qquad \frac{\partial \Theta}{\partial t} + \left(\frac{\partial U \theta^t}{\partial x} + \frac{\partial \Omega \theta^t}{\partial \eta} \right)^{\tau+\Delta\tau} = R_\Theta^t$$

$$\begin{matrix} W^{\tau+\Delta\tau} \\ \phi^{\tau+\Delta\tau} \end{matrix} \left\{ \begin{array}{l} \dfrac{\partial W}{\partial t} + g\overline{\left(\mu_d - \dfrac{\alpha}{\alpha_d} \dfrac{\partial p}{\partial \eta} \right)}^{\tau} = R_W^t \\ \mu_d^t \dfrac{\partial \phi}{\partial t} + U^{\tau+\Delta\tau} \dfrac{\partial \phi}{\partial x}^t + \Omega^{\tau+\Delta\tau} \dfrac{\partial \phi}{\partial \eta}^t - g\overline{W}^{\tau} = R_\phi^t \end{array} \right.$$

- Forward-backward differencing on U, Θ, and μ equations
- Vertically implicit differencing on W and φ equations

Dynamics: 4. Time integration scheme - implementation

Begin time step

Runge-Kutta loop (steps 1, 2, and 3)

(i) advection, p-grad, buoyancy using (φ^{t}, φ^{*}, φ^{**})

(ii) physics if step 1, save for steps 2 and 3

(iii) mixing, other non-RK dynamics, save…

(iv) assemble dynamics tendencies

Acoustic step loop

(i) advance U,V, then μ, Θ, then w, φ

(ii) time-average U,V, Ω

End acoustic loop

Advance scalars using time-averaged U,V, Ω

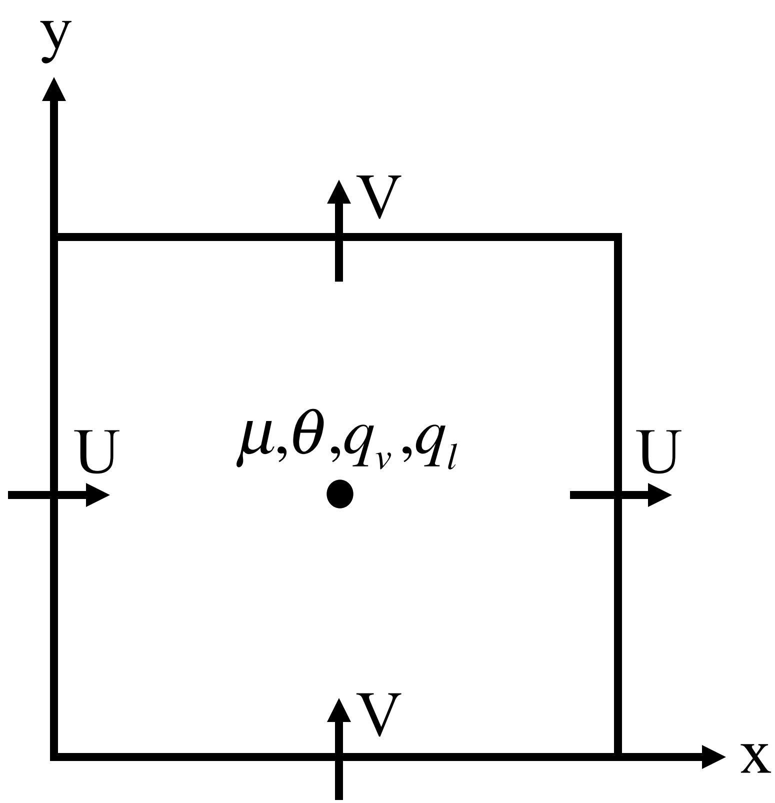
End Runge-Kutta loop

Adjustment physics (currently microphysics)

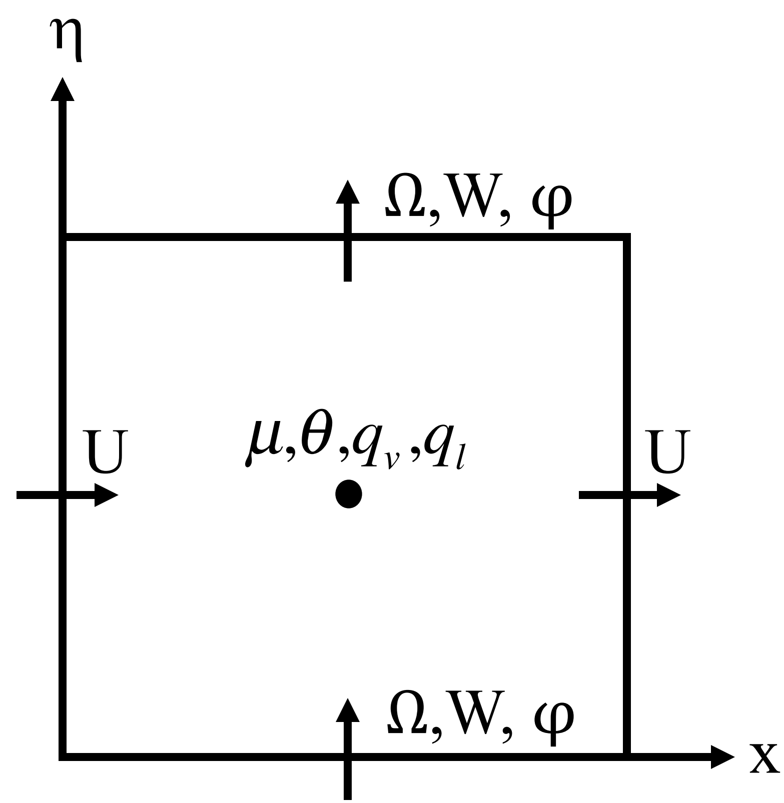
End time step

Dynamics: 5. Grid staggering – horizontal and vertical

C-grid staggering



horizontal



vertical

Dynamics: 6. Advection (transport) and conservation – dry-air mass

transport | pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d\frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d\frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta} + R_\theta + Q_\theta$$

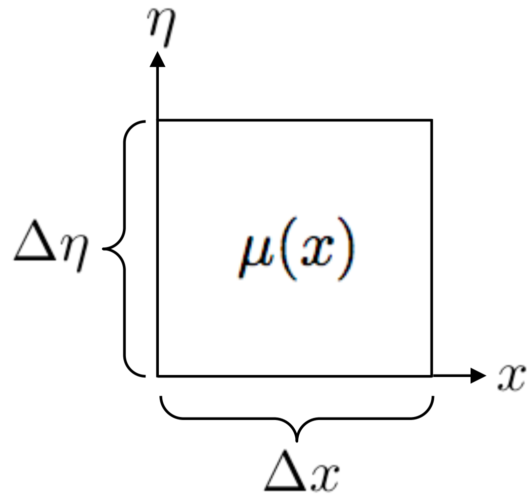
$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

Next:
Dry-air mass conservation in WRF

Diagnostic relations: $\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, p = \left(\frac{R_d \Theta_m}{p_o \mu_d \alpha_d}\right)^\gamma, \Theta_m = \Theta\left(1 + \frac{R_v}{R_d} q_v\right)$

control volume (2D example) →



Mass in a control volume is proportional to

$$(\Delta x \Delta \eta)(\mu)^t$$

since $\mu(x)\Delta\eta = \Delta\pi = -g\rho\Delta z$

Dynamics: 6. Advection (transport) and conservation – dry-air mass

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

2D example

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

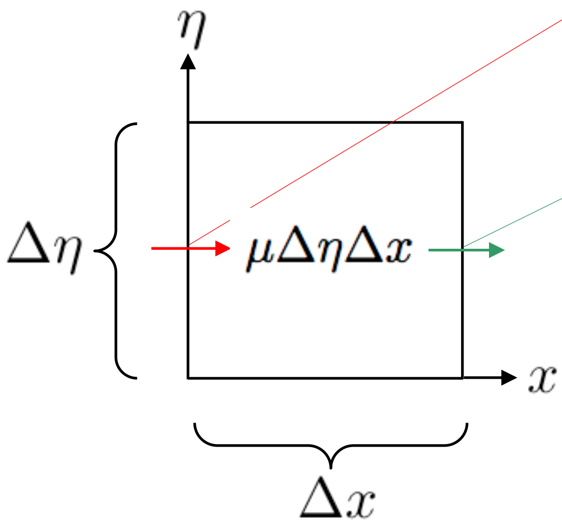
Change in mass over a time step

mass fluxes through control volume faces

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$



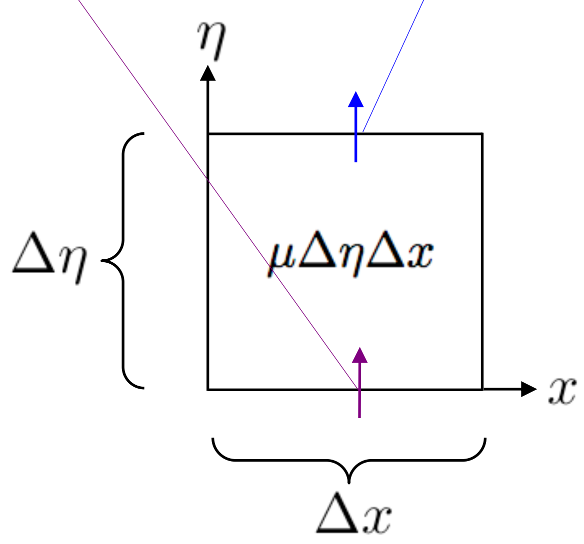
Horizontal fluxes through the vertical control-volume faces

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

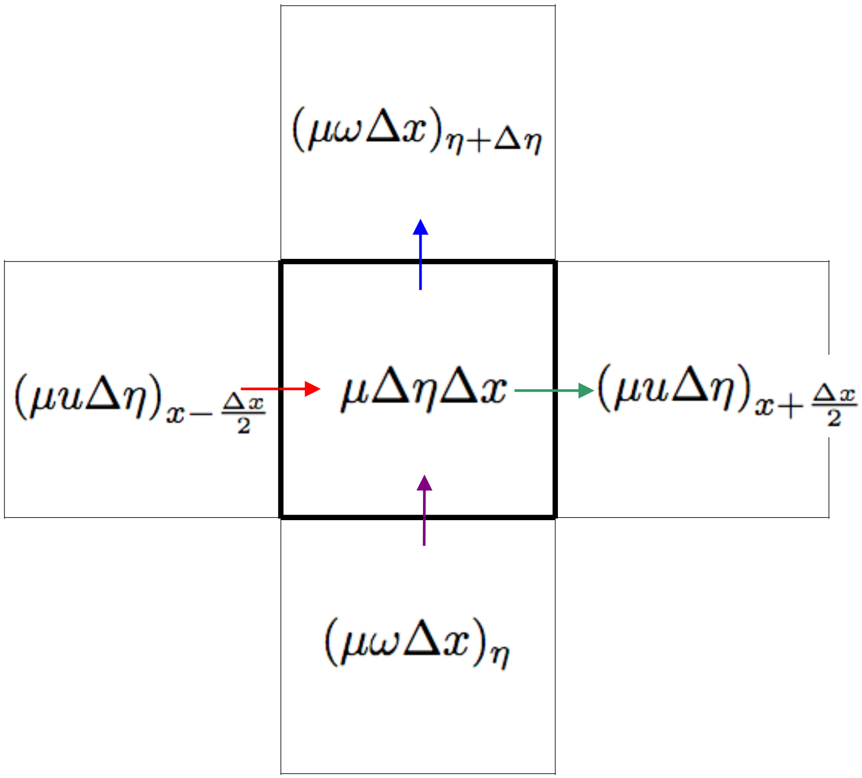
Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

Vertical fluxes through the horizontal control-volume faces



The same mass fluxes are used for neighboring grid cells - hence mass is conserved locally and globally.



Dynamics: 6. Advection (transport) and conservation

transport | pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d \frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta} + R_\theta + Q_\theta$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

Entropy and scalar mass conservation in WRF

Diagnostic relations: $\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, p = \left(\frac{R_d \Theta_m}{p_o \mu_d \alpha_d}\right)^\gamma, \Theta_m = \Theta\left(1 + \frac{R_v}{R_d} q_v\right)$

Mass in a control volume	$(\Delta x \Delta \eta)(\mu)^t$
Scalar mass	$(\Delta x \Delta \eta)(\mu\phi)^t$

Mass conservation equation:

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta\eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta\eta/2})\right]$$

change in mass over a time step

mass fluxes through control volume faces

Scalar mass conservation equation:

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu\phi)^{t+\Delta t} - (\mu\phi)^t)\right] = \left[(\mu u \phi \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \phi \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \phi \Delta x)_{x,\eta-\Delta\eta/2} - (\mu \omega \phi \Delta x)_{x,\eta+\Delta\eta/2})\right]$$

change in tracer mass over a time step

tracer mass fluxes through control volume faces

Dynamics: 6. Advection (transport) and conservation

transport — pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d\frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d\frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta} + R_\theta + Q_\theta$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

Transport schemes: flux divergence (transport) options in WRF

Diagnostic relations: $\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, \; p = \left(\frac{R_d \Theta_m}{p_o \mu_d \alpha_d}\right)^\gamma, \; \Theta_m = \Theta\left(1 + \frac{R_v}{R_d} q_v\right)$

2nd, 3rd, 4th, 5th and 6th order centered and upwind-biased schemes are available in the ARW model.

Example: 5th order scheme

$$\frac{\partial(U\psi)}{\partial x} = \frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(U\psi) - F_{i-\frac{1}{2}}(U\psi) \right)$$

where

$$F_{i-\frac{1}{2}}(U\psi) = U_{i-\frac{1}{2}}\left\{ \frac{37}{60}(\psi_i + \psi_{i-1}) - \frac{2}{15}(\psi_{i+1} + \psi_{i-2}) + \frac{1}{60}(\psi_{i+2} + \psi_{i-3}) \right\}$$

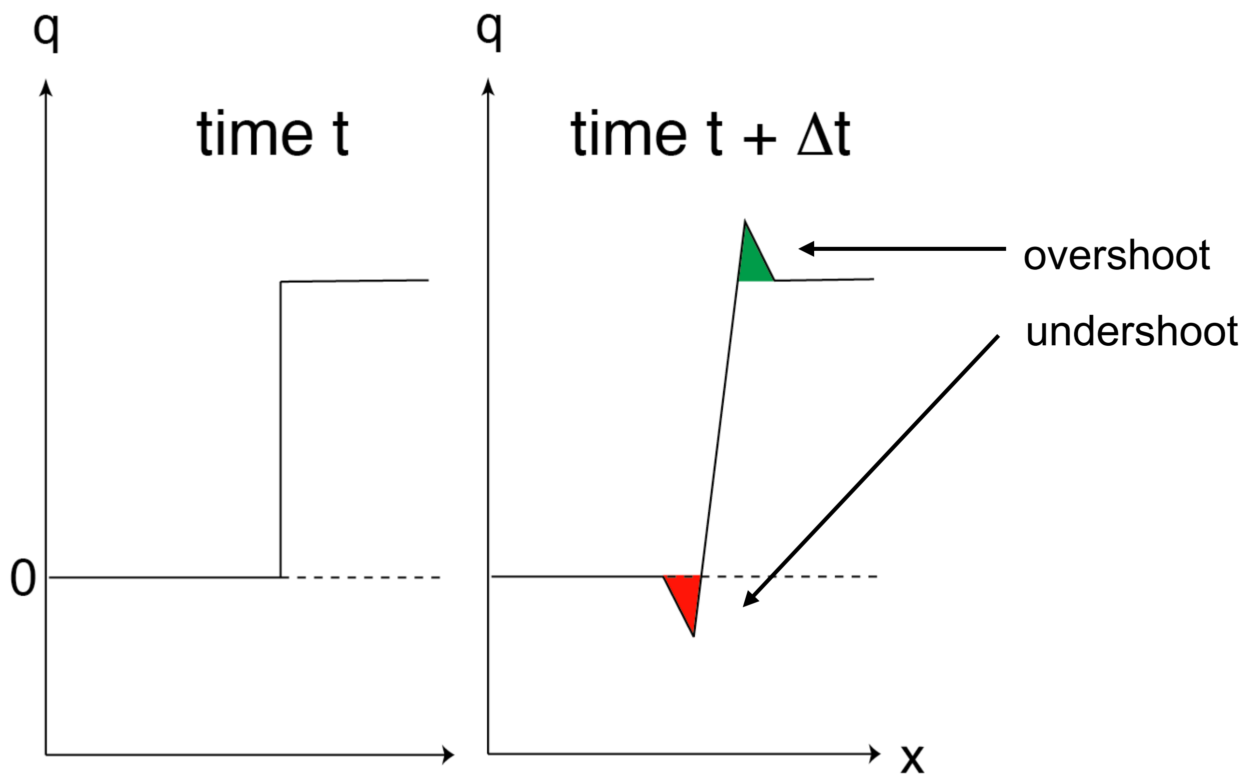
$$- sign(1,U)\,\frac{1}{60}\left\{ (\psi_{i+2} - \psi_{i-3}) - 5(\psi_{i+1} - \psi_{i-2}) + 10(\psi_i - \psi_{i-1}) \right\}$$

For constant U, the 5th order flux divergence tendency becomes

$$\Delta t \left.\frac{\delta(U\psi)}{\Delta x}\right|_{5th} = \Delta t \left.\frac{\delta(U\psi)}{\Delta x}\right|_{6th} - \underbrace{\left|\frac{U\Delta t}{\Delta x}\right| \frac{1}{60}\left(-\psi_{i-3} + 6\psi_{i-2} - 15\psi_{i-1} + 20\psi_i - 15\psi_{i+1} + 6\psi_{i+2} - \psi_{i+3}\right)}_{\frac{Cr}{60}\frac{\partial^6\psi}{\partial x^6} + H.O.T}$$

The odd-ordered flux divergence schemes are equivalent to the next higher ordered (even) flux-divergence scheme plus a dissipation term of the higher even order with a coefficient proportional to the Courant number.

1D advection



ARW transport is conservative,
but not positive definite nor shape preserving.
Removal of negative q
results in spurious source of q .

Dynamics: 6. Advection (transport) and conservation – shape preserving

Scalar update, last RK3 step

$$(\mu\phi)^{t+\Delta t} = (\mu\phi)^t - \Delta t \sum_{i=1}^{n} \delta_{x_i}[f_i] \qquad (1)$$

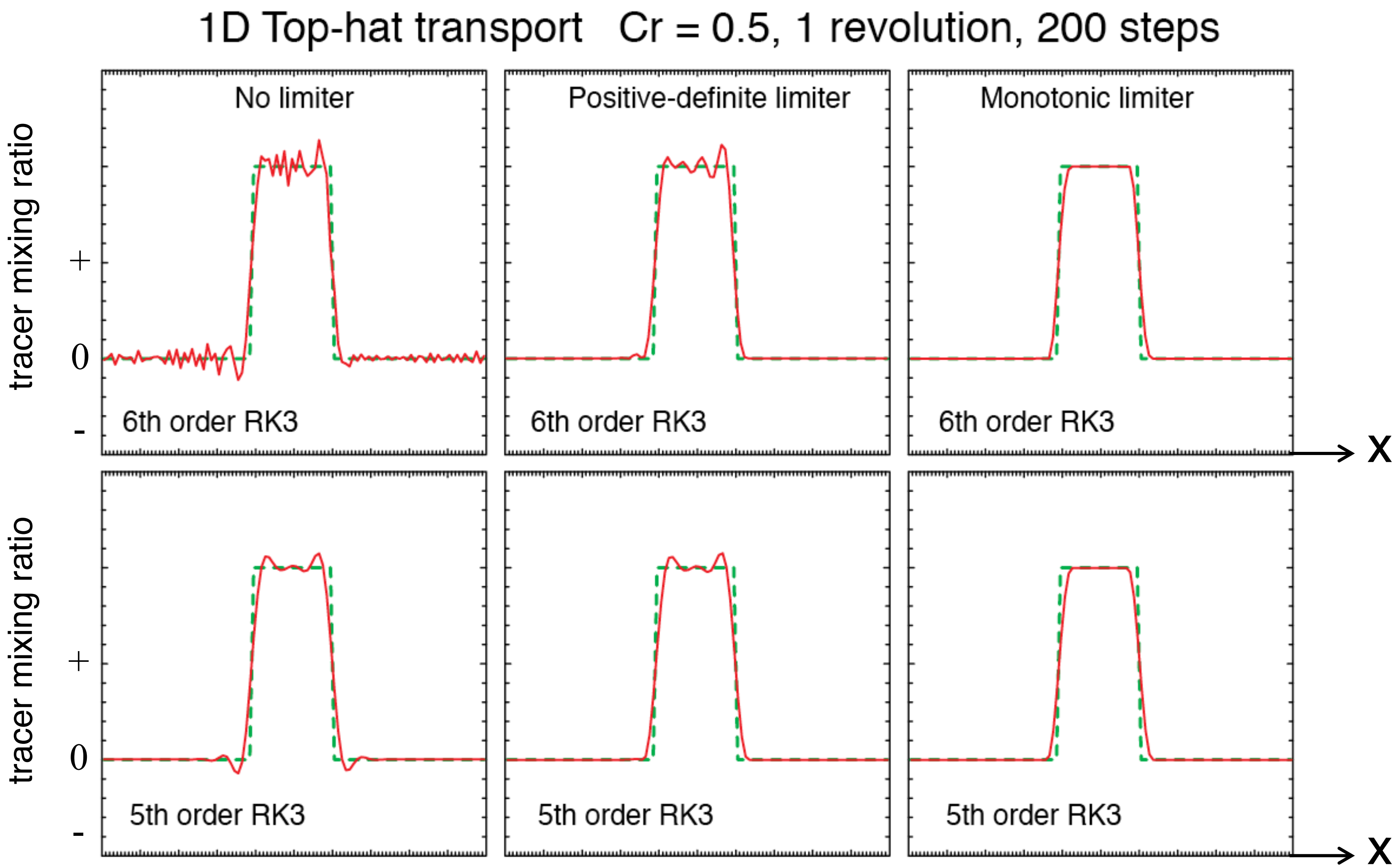
(1) Decompose flux: $f_i = f_i^{upwind} + f_i^c$

(2) Renormalize high-order correction fluxes f_i^c such that solution is positive definite or monotonic: $f_i^c = R(f_i^c)$

(3) Update scalar eqn. (1) using $f_i = f_i^{upwind} + R(f_i^c)$

This is a form of flux-correct transport (Zalesak 1979)

1D Example: Top-Hat Advection



Dynamics: 6. Advection (transport) and conservation

Where are the transport-scheme parameters?

The namelist.input file:

&dynamics

h_mom_adv_order
v_mom_adv_order
h_sca_adv_order
v_sca_adv_order

scheme order (2, 3, 4, 5 or 6)
defaults:
horizontal (h_*) = 5
vertical (v_*) = 3

momentum_adv_opt →

= 1 standard scheme
= 3 5^{th} order WENO
default: 1

moist_adv_opt
scalar_adv_opt
chem_adv_opt
tracer_adv_opt
tke_adv_opt

options:
= 1, 2, 3 : no limiter,
positive definite (PD),
montonic
= 4 : 5^{th} order WENO
= 5 : 5^{th} order PD WENO

Dynamics: 6. Advection (transport) and conservation

Where are the transport-scheme parameters?

The namelist.input file:
&dynamics

- The positive definite limiter (option 2) is enabled in the default WRF configuration.
- Chemistry applications typically use the monotonic limiter (option 3).
- Option 5 (PD WENO) is used in some applications employing multi-moment microphysics.

scheme order (2, 3, 4, 5 or 6)
defaults:
horizontal (h_*) = 5
vertical (v_*) = 3

= 1 standard scheme
= 3 5th order WENO
default: 1

moist_adv_opt
scalar_adv_opt
chem_adv_opt
tracer_adv_opt
tke_adv_opt

options:
= 1, 2, 3 : no limiter,
positive definite (PD),
montonic
= 4 : 5th order WENO
= 5 : 5th order PD WENO

3rd order Runge-Kutta time step
(the main timestep in WRF)

$$\Delta t_{RK}$$

Where? The namelist.input file:

&domains

time_step (integer seconds)

time_step_fract_num

time_step_fract_den

Guidelines for time step selection

Δt_{RK} in seconds should be about $6\Delta x$ (grid size in kilometers).

For example, for $\Delta x=10\ km$ choose $\Delta t=60$ seconds.

Dynamics: 7. Time step parameters

3rd order Runge-Kutta time step (the main timestep in WRF) Δt_{RK} (&domains *time_step*)

Δt_{RK} in seconds should be about $6\Delta x$ (grid size in kilometers).

Acoustic time step

2D horizontal Courant number limited:

$$\Delta \tau_{acoustic} = \Delta t_{RK} / \text{(number of acoustic steps)}$$

Where? The namelist.input file:

&dynamics

time_step_sound (an even integer > = 2)

The ARW default for *time_step_sound* is 4

Dynamics: 7. Time step parameters

3rd order Runge-Kutta time step Δt_{RK} (&domains *time_step*)

Acoustic time step [&dynamics *time_step_sound* (integer)]

Are the RK and/or acoustic timesteps too big?
If ARW blows up (aborts) quickly (possibly a dynamics instability), try:

(1) Decreasing Δt_{RK} (this also decreases $\Delta \tau_{acoustic}$), or
(2) increasing the integer time_step_sound
(this decreases $\Delta \tau_{acoustic}$ but does not change Δt_{RK}).

Dynamics: 8. Filters – divergence damping

Purpose: filter acoustic modes (3-D divergence, $D = \nabla \cdot \rho \mathbf{V}$ *)*

$$\left\{ \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla p + \ldots = \gamma_d' \nabla D \right\}$$

$$\nabla \cdot \left\{ \quad \right\} \quad \rightarrow \quad \frac{\partial D}{\partial t} + \nabla^2 p + \ldots = \gamma_d' \nabla^2 D$$

From the pressure equation: $p_t \simeq c^2 D$

$$\boxed{\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla [p_\tau + \gamma_d (p^\tau - p^{\tau - \Delta \tau})] + \ldots = 0}$$

$\gamma_d = 0.1$ recommended (default) (&dynamics *smdiv*)

(Illustrated in height coordinates for simplicity)

Dynamics: 8. Filters – time off-centering the vertical acoustic modes

Purpose: damp vertically-propagating acoustic modes

$$\frac{\partial W}{\partial t} + g\overline{\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right)}^{\tau} = \ldots$$

$$\frac{\partial \phi}{\partial t} - \frac{g}{\mu_d^t}\overline{W}^{\tau} = \ldots$$

$$\overline{(\quad)}^{\tau} = \frac{1+\beta}{2}\overline{(\quad)}^{\tau+\Delta\tau} + \frac{1-\beta}{2}\overline{(\quad)}^{\tau}$$

Slightly forward centering the vertical pressure gradient damps 3-D divergence as demonstrated for the divergence damper

$\beta = 0.1$ recommended (default) [&dynamics *epssm*]

Purpose: filter the external mode

Vertically integrated horizontal divergence, $D_h = \int_1^0 (\nabla_\eta \cdot \mu \mathbf{V}_h) d\eta$

$$\left\{ \frac{\partial \mu \mathbf{V}_h}{\partial t} + \ldots = -\gamma_e \nabla_\eta D_h \right\}$$

$$\int_1^0 \nabla_\eta \cdot \left\{ \quad \right\} d\eta \quad \rightarrow \quad \frac{\partial D_h}{\partial t} + \ldots = \gamma_e \nabla^2 D_h$$

Continuity equation: $\frac{\partial \mu}{\partial t} = -\nabla_\eta \cdot \mu \mathbf{V}_h - \frac{\partial \mu \dot{\eta}}{\partial \eta} = D_h$

$$\boxed{\frac{\partial \mu \mathbf{V}_h}{\partial \tau} + \ldots = -\gamma_e \frac{\Delta x^2}{\Delta \tau} \nabla_\eta (\mu^\tau - \mu^{\tau - \Delta\tau})}$$

$\gamma_e = 0.01$ recommended (default) [&dynamics *emdiv*]

(Primarily for real-data applications)

Purpose: damp anomalously-large vertical velocities

(usually associated with anomalous physics tendencies)

Additional term:

$$\partial_t W = \ldots \underline{- \mu_d \, \mathrm{sign}(W) \gamma_w (Cr - Cr_\beta)}$$

$$Cr = \left| \frac{\Omega dt}{\mu d\eta} \right|$$

$Cr_\beta = 1.0$ typical value (default)

[share/module_model_constants.F *w_beta*]

$\gamma_w = 0.3$ m/s^2 recommended (default)

[share/module_model_constants.F *w_alpha*]

[&dynamics *w_damping* 0 (off; default) 1 (on)]

2nd-Order Horizontal Mixing, Horizontal-Deformation-Based K_h

Purpose: mixing on horizontal coordinate surfaces (real-data applications) [&dynamics *diff_opt=1, km_opt=4*]

$$K_h = C_s^2\, l^2 \left[0.25(D_{11} - D_{22})^2 + \overline{D_{12}^2}^{xy} \right]^{\frac{1}{2}}$$

where $l = (\Delta x \Delta y)^{1/2}$

$$D_{11} = 2\, m^2 \left[\partial_x(m^{-1}u) - z_x \partial_z(m^{-1}u) \right]$$
$$D_{22} = 2\, m^2 \left[\partial_y(m^{-1}v) - z_y \partial_z(m^{-1}v) \right]$$
$$D_{12} = m^2 \left[\partial_y(m^{-1}u) - z_y \partial_z(m^{-1}u) + \partial_x(m^{-1}v) - z_x \partial_z(m^{-1}v) \right]$$

$C_s = 0.25$ (Smagorinsky coefficient, default value) [&dynamics *c_s*]

Implicit Rayleigh w Damping Layer for Split-Explicit Nonhydrostatic NWP Models (gravity-wave absorbing layer)

$$W^{\tau+\Delta\tau} = W^{*\tau+\Delta\tau} - \Delta\tau R_w(\eta) W^{\tau+\Delta\tau}$$

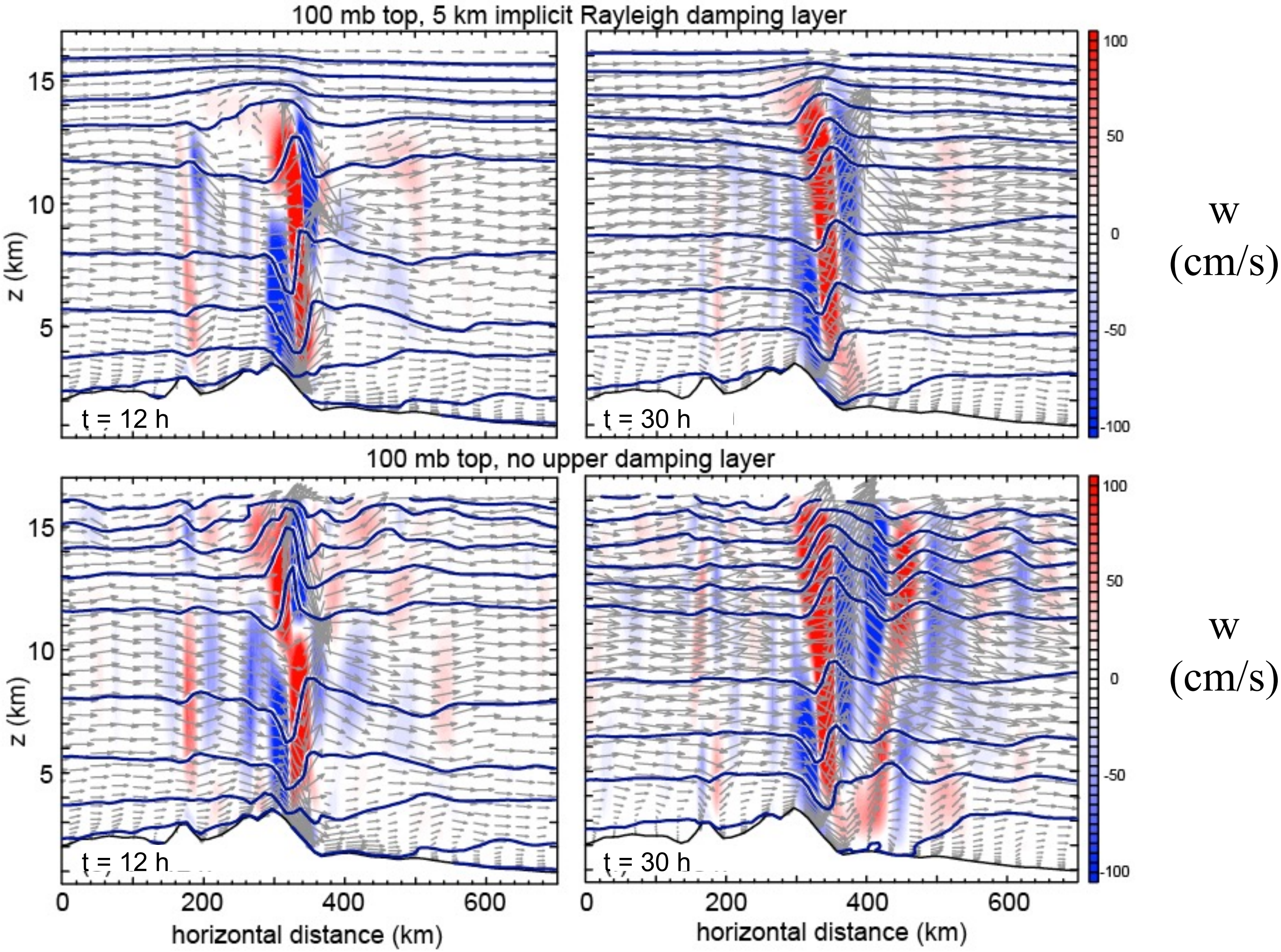
$$R_w(\eta) = \begin{cases} \gamma_r \sin^2\left[\frac{\pi}{2}\left(1 - \frac{z_{top}-z}{z_d}\right)\right] & \text{for } z \geq (z_{top} - z_d); \\ 0 & \text{otherwise,} \end{cases}$$

$R_w(\eta)$- damping rate (t^{-1})
z_d - depth of the damping layer
γ_r - damping coefficient

[&dynamics *damp_opt = 3* (default = 0)]
[&dynamics *damp_coef = 0.2* (recommended, = 0. default)]
[&dynamics *zdamp = 5000.* (z_d *(meters)*; default); height below model top where damping begins]

Dynamics: 8. Filters – gravity-wave absorbing layer example

Model Initialized 04 Dec 2007 00 UTC

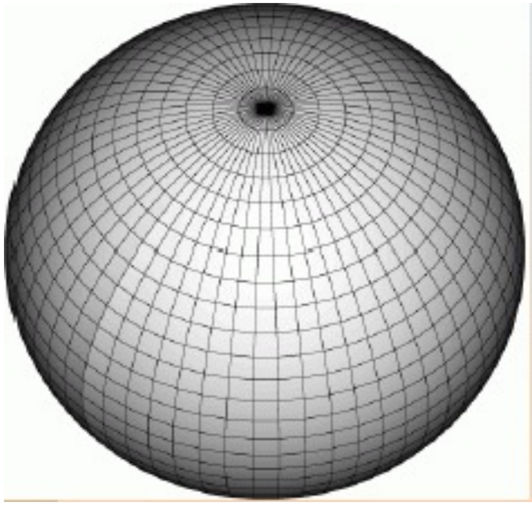


ARW Model: projection options

1. Cartesian geometry:
   idealized cases
2. Lambert Conformal:
   mid-latitude applications
3. Polar Stereographic:
   high-latitude applications
4. Mercator:
   low-latitude applications
5. Latitude-Longitude global, regional

Projections 1-4 are isotropic ($m_x = m_y$)
Latitude-longitude projection is anistropic ($m_x \neq m_y$)

Global ARW – Polar filters

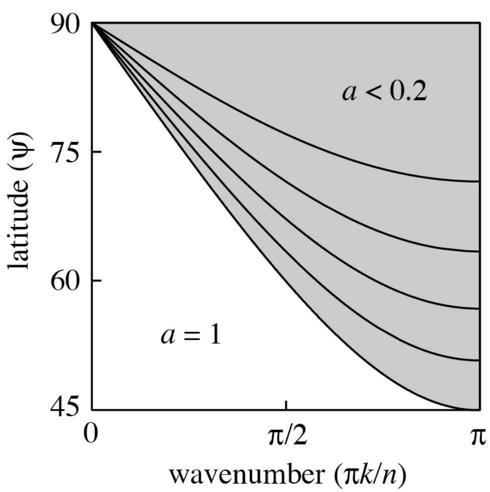


Converging gridlines severely limit timestep.
The polar filter removes this limitation.

Filter procedure - Along a grid latitude circle:

1. Fourier transform variable.
2. Filter Fourier coefficients.
3. Transform back to physical space.

Filter Coefficient $a(k)$, $\psi_o = 45^o$



$$\hat{\phi}(k)_{filtered} = a(k)\,\hat{\phi}(k), \quad \text{for all } k$$

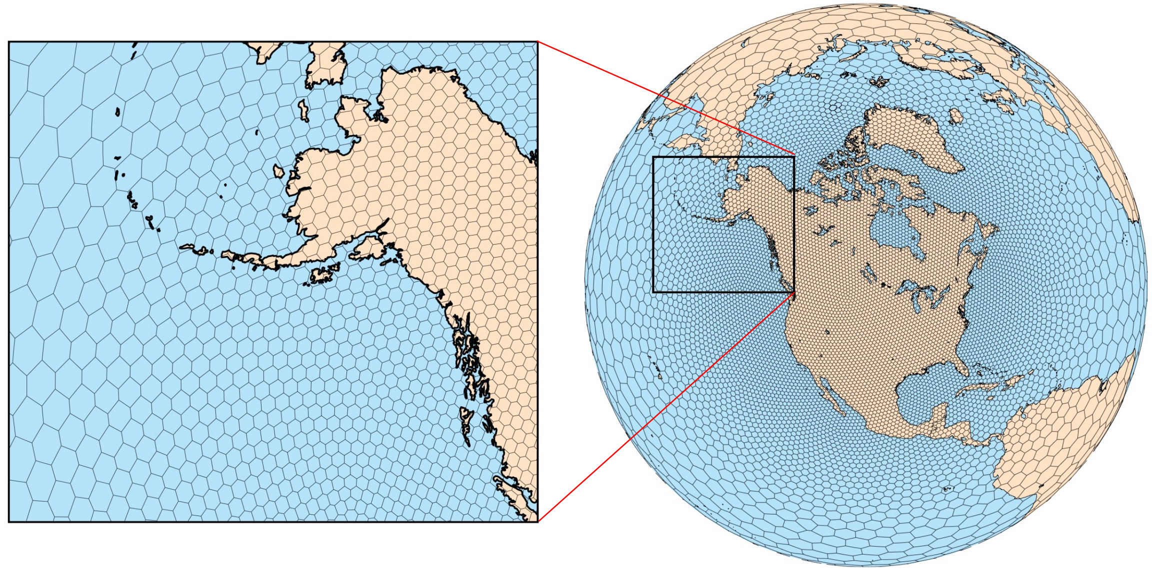
$$a(k) = \min\left[1., \max\left(0., \left(\frac{\cos\psi}{\cos\psi_o}\right)^2 \frac{1}{\sin^2(\pi k/n)}\right)\right]$$

$k =$ dimensionless wavenumber
$\hat{\phi}(k) =$ Fourier coefficients from forward transform
$a(k) =$ filter coefficients
$\psi =$ latitude $\psi_o =$ polar filter latitude, filter when $|\psi| > \psi_o$

Dynamics: 9. Map projections and global configuration

An alternative to global ARW…

MPAS
Model for Prediction Across Scales



- Global, nonhydrostatic, C-grid Voronoi mesh
- Numerics similar to WRF; WRF-NRCM physics
- No pole problems
- Variable-resolution mesh – no nested BC problems
- Regional capability

Available at: http://mpas-dev.github.io/

ARW Model: Boundary Condition Options

Lateral boundary conditions

1. Specified (Coarse grid, real-data applications).
2. Open lateral boundaries (gravity-wave radiative).
3. Symmetric lateral boundary condition (free-slip wall).
4. Periodic lateral boundary conditions.
5. Nested boundary conditions (specified).

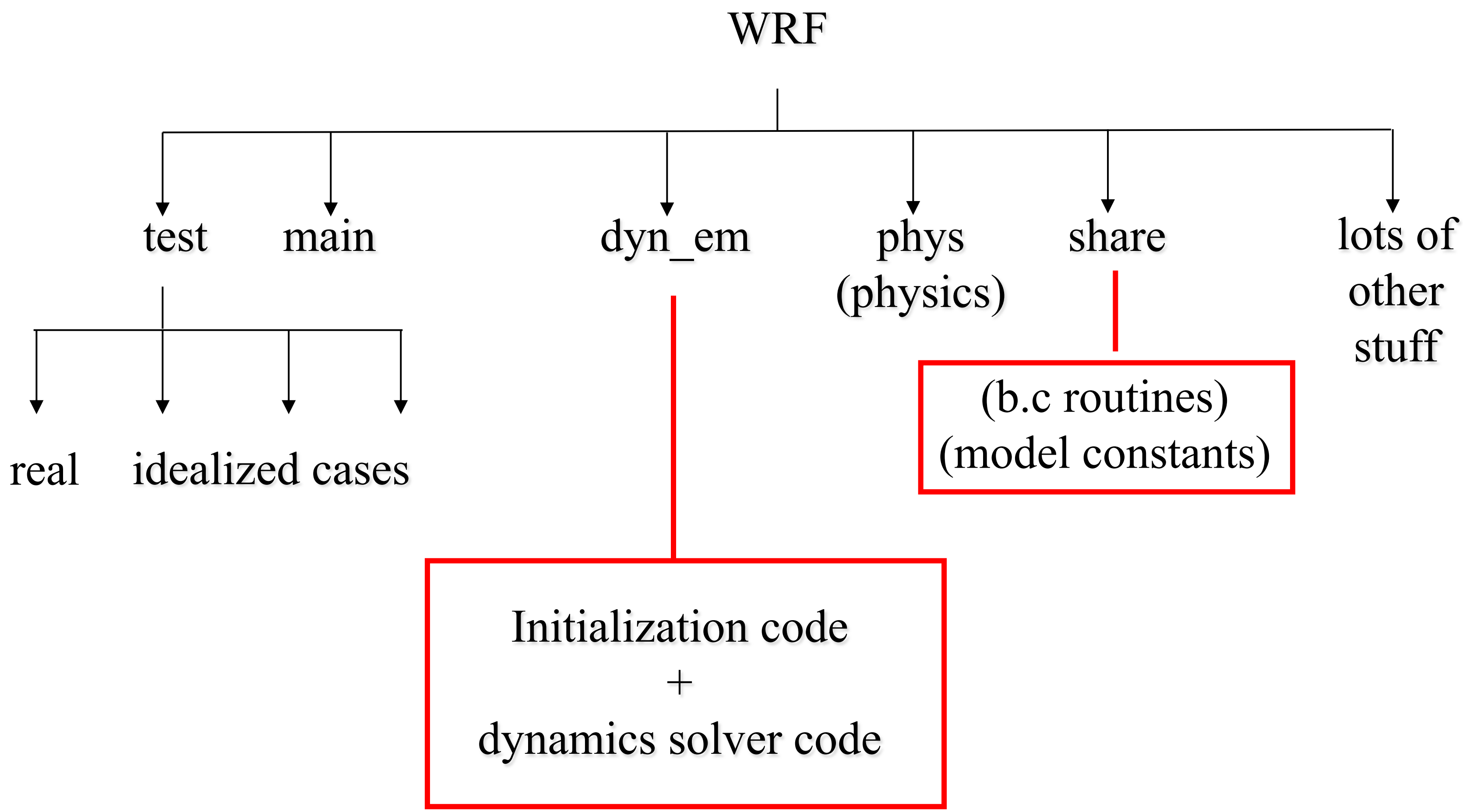
Top boundary conditions

1. Constant pressure.

Bottom boundary conditions

1. Free slip.
2. Various B.L. implementations of surface drag, fluxes.

Dynamics: Where are things?



WRF ARW Tech Note

A Description of the Advanced Research WRF Version 4 (March 2021 ; WRF Version 4.3)

http://www2.mmm.ucar.edu/wrf/users/docs/technote/contents.html