The Mathematical Representation of Wind Speed and Temperature Profiles in the Unstable Atmospheric Surface Layer[1]

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ABSTRACT

Analytical expressions which specify non-dimensionalized wind speed and potential temperature gradients as functions of stability are integrated. The integrated equations are tested against Swinbank's wind and temperature profiles measured at Kerang, Australia. It is found that a representation suggested independently by Businger and by Dyer gives the best fit to temperature profiles and describes the wind profiles equally as well as a relation suggested by Panofsky *et al.*

1. Introduction

The similarity hypothesis of Monin and Obukhov (1954) has been widely used as a basis for the description of mean wind speed and temperature as a function of height in the atmospheric surface layer, defined as the layer near the earth's surface in which the turbulent fluxes are approximately constant with height. Lumley and Panofsky (1964) have summarized the similarity hypothesis and associated results. For a dry atmosphere, we have the prediction that the non-dimensional wind speed and potential temperature gradients, ϕ_1 and ϕ_2, defined by

$$\left.\begin{aligned}\phi_1&=\frac{kz}{u_*}\frac{\partial u}{\partial z}\\ \phi_2&=\frac{z}{\theta_*}\frac{\partial \theta}{\partial z}\end{aligned}\right\}, \qquad (1)$$

should be universal functions of a non-dimensional stability parameter, such as z/L, z/L' or Ri, where:

L	$-u_*^3 c_p \rho T/(kgH)$
u	mean wind speed
z	vertical space coordinate
T	mean temperature
k	von Kármán's constant ($=0.4$)
u_*	$(\tau/\rho)^{1/2}$, friction velocity
H	$c_p\rho\overline{W'T'}$, turbulent heat flux
θ_*	$-(1/ku_*)(H/\rho c_p)$, a scaling temperature
θ	$T+\Gamma z$, potential temperature
Γ	g/c_p, adiabatic lapse rate
g	acceleration due to gravity
c_p	specific heat at constant pressure
ρ	air density
τ	tangential stress
L'	αL
Ri	$(g/T)[(\partial\theta/\partial z)/(\partial u/\partial z)^2]$, gradient Richardson number
α	K_h/K_m, reciprocal of the turbulent Prandtl number
K_m	$u_*^2/(\partial u/\partial z)$, turbulent transfer coefficient for momentum
K_h	$-H/[\rho c_p(\partial\theta/\partial z)]$, turbulent transfer coefficient for heat

The forms of the universal functions ϕ_1 and ϕ_2, besides being of interest in themselves, are of practical importance for diverse purposes, among them the estimation of turbulent fluxes, the prediction of diffusion of pollutants, and the estimation of wind force on structures.

The purpose of this paper is to show how previously suggested functions may be integrated and to test the integrated functions against profile observations taken at Kerang, Australia (Swinbank, 1964) during unstable stratification. The method of analysis used is also suitable for estimation of turbulent fluxes from profiles and is similar to that suggested by Panofsky (1963).

2. Profile representations

The non-dimensional gradients, ϕ_1 and ϕ_1, may be integrated, following Panofsky (1963), in the form

$$\left.\begin{aligned}u&=\frac{u_*}{k}[\ln(z/z_0)-\psi_1]\\ \theta-\theta_0&=\theta_*[\ln(z/z_0)-\psi_2]\end{aligned}\right\}, \qquad (2)$$

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where z_0 is the roughness length and

$$\left.\begin{aligned}\psi_1&=\int_0^{\xi}\frac{1-\phi_1(\xi')}{\xi'}d\xi'\\ \psi_2&=\int_0^{\xi}\frac{1-\phi_2(\xi')}{\xi'}d\xi'\end{aligned}\right\},\tag{3}$$

where ξ is equal to either z/L or z/L', depending on whether ϕ_1 and ϕ_2 are taken as functions of z/L or z/L'.

An equation which interpolates between neutral stability and free convection,

$$\phi_1^4-\gamma(z/L')\phi_1^3=1,\tag{4}$$

was suggested by Panofsky *et al.* (1960), where γ is a constant determined from observations. The function ψ_1, defined in (3), may be transformed from a function of z/L' to a function of ϕ_1 by use of (4), which becomes, after arranging the integrand into easily integrated partial fractions,

$$\psi_1=\int_0^{\phi_1}\left(-1+\frac{2}{1+\phi_1'}+\frac{2}{1+\phi_1'^2}+\frac{2\phi_1'}{1+\phi_1'^2}-\frac{3}{\phi_1'}\right)d\phi_1'.$$

Carrying out the integration, we obtain

$$\psi_1=1-\phi_1-3\ln\phi_1+2\ln[(1+\phi_1)/2]+2\tan^{-1}\phi_1-\pi/2+\ln[(1+\phi_1^2)/2].\tag{5}$$

If we assume that α (the reciprocal of the turbulent Prandtl number) is constant and equal to unity, we have $\phi_2=\phi_1$, and therefore, $\psi_2=\psi_1$. It follows from the definitions and (4) that

$$\frac{z}{L'}=\frac{\mathrm{Ri}}{(1-\gamma\,\mathrm{Ri})^{1/4}}.\tag{6}$$

With Ri computed from profile measurements, one may then determine z/L' from (6), ϕ_1 from (4), and finally, ψ_1 from (5). The solution of (4) for ϕ_1 with z/L' given may be done by trial and error procedure on an electronic computer (straightforward since $0\leqslant\phi_1\leqslant1$) or by the use of tables. Following Panofsky (1963) we shall refer to the model outlined above as the KEYPS representation.

Yamamoto (1959) has also integrated (4) as a function of ϕ_1, but his result has the disadvantage of a singular point at neutral conditions.

Businger (1966) and Dyer (unpublished) have independently suggested a profile representation for unstable conditions (hereafter referred to as B-D) which has as its basic elements the interpolation equation (4) and the hypothesis that $\mathrm{Ri}=z/L$. This hypothesis is based on the Kerang observations (Swinbank, 1964) and was also suggested by Pandolfo (1966) on the basis of his analysis of the same observations. The hypothesis leads to the relation

$$\alpha=1/\phi_1.$$

Hence, the model has $\alpha=1$ for neutral conditions and increasing for decreasing stability, in agreement with a suggestion by Priestly and Swinbank (1947). The B-D representation may be summarized by the set of equations:

$$\left.\begin{aligned}\mathrm{Ri}&=z/L\\ \phi_1&=[1-\gamma(z/L)]^{-1/4}\\ \phi_2&=[1-\gamma(z/L)]^{-1/2}\end{aligned}\right\}.$$

Carrying out the integration (3) to obtain ψ_1 and ψ_2, we have

$$\left.\begin{aligned}\psi_1&=2\int_1^x\left(\frac{1}{1+x'}+\frac{x'}{1+x'^2}-\frac{1}{1+x'^2}\right)dx'\\ \psi_1&=2\ln[(1+x)/2]+\ln[(1+x^2)/2]-2\tan^{-1}x+\pi/2\\ \psi_2&=2\int_1^{x^2}\frac{d(x'^2)}{1+x'^2}\\ \psi_2&=2\ln[(1+x^2)/2]\end{aligned}\right\},$$

where

$$x=(1-\gamma z/L)^{1/4}.$$

It should be noted, for $-z/L$ large, that the B-D representation implies $u\propto z^{-1/4}$ and $\theta\propto z^{-1/2}$, which may be contrasted with the KEYPS representation where both u and θ vary asymptotically as $z^{-1/3}$.

A third representation that we shall consider is the log-linear with z/L' as the stability parameter. We shall assume $\alpha=1$ with the result that $\phi_1=\phi_2$ and $\psi_1=\psi_2$. The representation may be obtained by expanding (4) for small z/L', yielding

$$\left.\begin{aligned}\phi_1&=1+(\gamma/4)(z/L')\\ \psi_1&=-(\gamma/4)(z/L')\end{aligned}\right\},$$

where once again γ is a constant determined from observations. The relation between Ri and z/L' for this case is

$$\frac{z}{L'}=\frac{\mathrm{Ri}}{1-(\gamma/4)\mathrm{Ri}}.$$

3. Method of analysis

First, Ri is computed for a particular run by use of the lowest and highest observations. Derivatives are approximated by

$$\left.\frac{\partial F}{\partial z}\right|_{(z_1z_2)^{1/2}}\approx\frac{F_2-F_1}{(z_1z_2)^{1/2}}\ln(z_2/z_1),$$

following a suggestion of Panofsky (1965), where F is a profile variable. The approximation applies at the geometric mean height of the observation levels and is rigorous for log profiles (neutral conditions). For $F \propto z^{-1/3}$ with observation heights of 0.5 and 16 m (Kerang data), the error is $\sim 6\%$. This error is not significant when compared to the present uncertainty of the profile formulas, but if desired, it could be reduced by use of a correction based on the particular representation under consideration.

In the next step of the analysis, Ri is used to determine L and L' by use of the appropriate relationship for each of the models. One may then compute ψ_1 and ψ_2 for each of the observation levels. Eq. (2) for u may be written in the form

$$u = (u_*/k)(\ln z - \psi_1) - (u_*/k)\ln z_0.$$

A similar equation follows for $\theta - \theta_0$. Hence, if a particular representation is correct, observations of u and θ plotted vs $\ln z - \psi_1$ and $\ln z - \psi_2$, respectively, will be linear. Therefore, we fit a straight line by the method of least squares to u vs $\ln z - \psi_1$, which yields an estimate of u_* and z_0. Carrying out a similar procedure for θ yields an estimate of θ_* from which one may compute the heat flux.

The variance of the differences between the observations and the fitted curve is computed to give an estimate of how well the representation (including a particular choice of the arbitrary constant) fits the data. For a series of runs, the variance is averaged over the entire series in order to compare how well different representations fit the entire set of observations.

It should be noted that the analysis can be extended to include humidity profiles measured simultaneous to those of wind and temperature. An example of such an extension is given by Paulson *et al.* (1970).

4. Results and discussion

The analysis described above has been carried out on a set of 34 observations of mean wind and temperature profiles reported by Swinbank (1964). These observations are acknowledged to be of very good quality, both as regards uniformity of the site and care taken in making the measurements.

All of the runs were taken during unstable stratification with observation heights ranging from 0.5–16 m. The terrain was dry.

The variance of the difference between the observations and the best fitting profiles is plotted in Fig. 1 for the different representations as a function of the arbitrary constant in each representation. The most striking result is that the B-D representation fits the temperature profiles much better than the other representations in which α is assumed constant. This result is in accordance with observations (e.g., Swinbank, 1964) which show α increasing from a value near 1 for neutral stability to values as large as 2–3 for very unstable conditions.

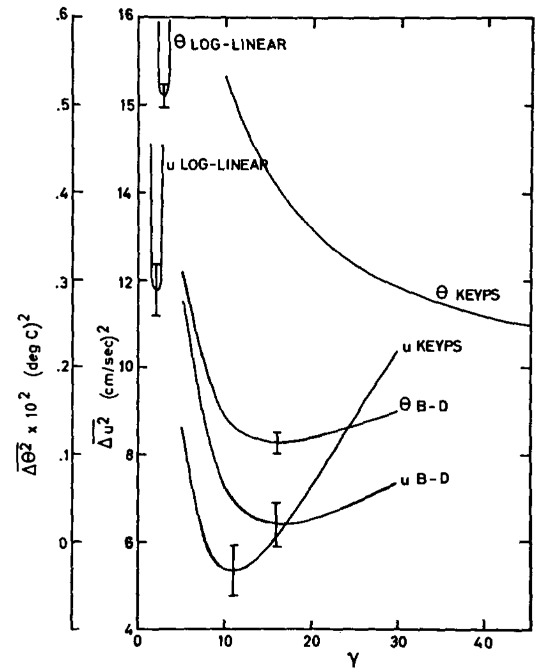

FIG. 1. Various mathematical representations to the Kerang profiles as a function of γ (the arbitrary constant in the representation). The vertical bars drawn at the best fitting gammas represent the standard error in the variance estimates computed by taking the variance of the difference between observations and the best fitting representations as the measure of uncertainty of the observations.

The poor fit of the log-linear representation to the observations is consistent with previous criticism (e.g., Taylor, 1960) that it is valid only in near-neutral conditions.

The Businger-Dyer and KEYPS representations appear to fit the wind observations equally well. The constant in the KEYPS equation which gives the best fit is 11, somewhat smaller than $\gamma = 18$ suggested by Panofsky *et al.* (1960). However, there is considerable uncertainty in the present estimate since the goodness of fit changes slowly with changing γ.

The B-D representation gives the best fit to both the wind and temperature profiles for $\gamma = 16$. The value 16 agrees well with

$$\phi_2 = (1 - 15z/L)^{-0.55},$$

suggested by Dyer (1967) on the basis of an analysis of several sets of observations including the set considered here.

Another method of testing the validity of profile representations is to plot $\ln z - \psi_1$ vs u and $\ln z - \psi_2$ vs θ as suggested by Panofsky (1963). This is done in Figs. 2 and 3 for the B-D representation for profiles averaged together according to stability classes given in Table 1. There is little evidence of any systematic departure from the straight lines drawn by eye to fit the observations.

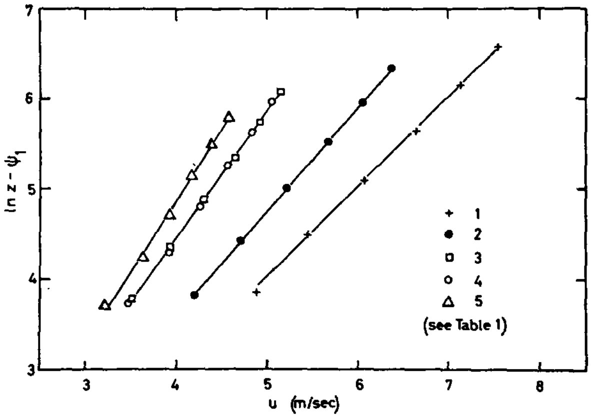

FIG. 2. Composite Kerang wind speeds as a function of $\ln z - \psi_1$ (B-D representation with $\gamma = 16$).

5. Conclusions

The Businger-Dyer representation gives the best simultaneous fit to both the wind speed and temperature profiles. The KEYPS representation is equally good in describing the wind profile, but with the assumption of α equal to a constant, it fits the temperature profile poorly. The log-linear representation does not fit the data well, consistent with previous suggestions that it should only apply for near-neutral stability.

The importance of knowledge of the behavior of α as a function of stability in obtaining a correct description of the temperature profile ought to encourage further experimental and theoretical investigation of this behavior. Of the representations considered, only that of B-D allowed any variation of α with stability, and although the specified behavior is at least qualitatively correct, it should not be considered definitive.

The uncertainty of the best fitting constants in the B-D and KEYPS representations indicates the necessity of having independent estimates of τ and H to determine these constants more precisely and also to provide a more critical test of the representations. It would be desirable to have simultaneous profile and flux measurements for widely different sites to check for

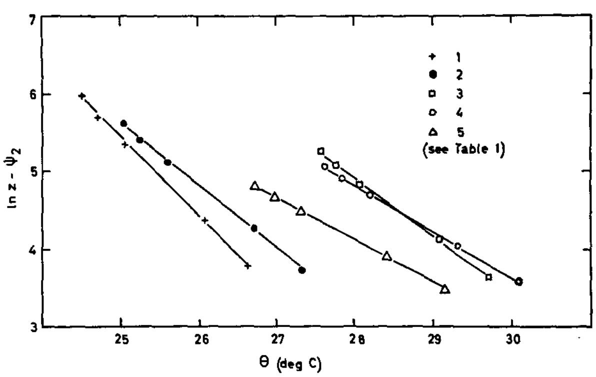

FIG. 3. Composite Kerang potential temperatures as a function of $\ln z - \psi_2$ (B-D representation with $\gamma = 16$).

TABLE 1. Grouping of Kerang runs according to Richardson number computed from the 0.5 and 16 m observations.

Group no.	Number of runs	Ri[1]
1	7	0.07–0.12
2	7	0.13–0.18
3	7	0.19–0.27
4	7	0.28–0.33
5	6	0.34–0.49

[1] All values negative.

possible effects of differences in boundary conditions, both upper and lower, or other parameters not taken into account. The good agreement reported by Miyake *et al.* (1970) between direct measurements of stress and heat flux with profile estimates determined by use of the Businger-Dyer model with $\gamma = 16$ gives further support to the validity of this model.

The practical utility of an accurate profile representation should be emphasized. Even though direct flux measurements are possible, they may not be as representative as profile estimates. Simultaneous direct measurements of heat flux by two instruments, reported by Businger *et al.* (1967), differed by as much as a factor of 2 for a horizontal separation of 5 m. Since profile measurements are samples in space as well as time, they may yield more representative flux estimates. Finally, one may use a profile model as the basis for a bulk parameterization of the fluxes along the lines suggested by Deardorff (1968) and Paulson (1969), whereby measurements of mean wind speed and temperature at a single height in the surface layer and a knowledge of z_0 and surface temperature would permit computation of H and τ.

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