

WRF Implementation Details and Version history of a Stochastic Kinetic-Energy Backscatter Scheme (SKEBS)

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1. Introduction

The aim of this note is to give details of the SKEBS implementation in WRF, which was first released as part of WRFV3.3. After the introduction, the second section derives the streamfunction and potential temperature perturbations. Details of version history and recommended namelist parameter values are given for each model release in section 3.

2. The stochastic kinetic-energy backscatter scheme (SKEBS)

This section describes details of the SKEBS implementation.

a. *Definition of energies*

Define a perturbation kinetic energy as

$$E_{\text{kin}} = \frac{1}{2}(u^2 + v^2) \quad (1)$$

and a perturbation potential energy as

$$E_{\text{pot}} = \frac{1}{2} \frac{c_p}{\theta_0} \theta^2 \quad (2)$$

where θ is the potential temperature, $c_p = 1004 \frac{J}{K}$ the specific heat capacity and $\theta_0 = 300K$ a reference potential temperature.

We assume that a fraction of the dissipated kinetic and potential energy is available as forcing for the resolved flow leading to streamfunction tendency and temperature tendency forcings. The backscattered fractions are assumed to be different for each energy component (see Shutts (2005) for a discussion). In the following we will derive the perturbations for streamfunction and potential temperature separately. Note that by perturbing streamfunction, we assume that the backscattered kinetic energy only affects the rotational component of the wind. This assumption is consistent with Shutts and Grey (1994), but depending on the application, one might want to force divergent wind as well (e.g., Bowler et al., 2009).

b. *Derivation for streamfunction tendency perturbations*

Let $\psi'(x, y, t)$ be a 2D streamfunction tendency forcing, $\psi'(x, y, t) = \frac{\partial \psi}{\partial t}$, and $u'(x, y, t)$ and $v'(x, y, t)$ the corresponding zonal and meridional wind tendency forcings expressed in 2D Fourier space:

$$\psi'(x, y, t) = \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \psi'_{k,l}(t) e^{2\pi i(kx/X+ly/Y)} \quad (3)$$

$$u'(x, y, t) = -\frac{\partial \psi'(x, y, t)}{\partial y} = -\frac{2\pi i}{Y} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} l \psi'_{k,l}(t) e^{2\pi i(kx/X+ly/Y)} \quad (4)$$

$$v'(x, y, t) = \frac{\partial \psi'(x, y, t)}{\partial x} = \frac{2\pi i}{X} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} k \psi'_{k,l}(t) e^{2\pi i(kx/X+ly/Y)}, \quad (5)$$

where k and l denote the $(K+1)(L+1)$ wavenumber components in the zonal x - and meridional y -direction in physical space and t denotes time. The Fourier modes $e^{2\pi i(kx/X+ly/Y)}$ form an orthogonal set of basis functions on the rectangular domain $0 < x < X$ and $0 < y < Y$. If the $\psi'_{k,l}$ are non-vanishing for at least one $|k| < \frac{K}{2}$ or $|l| < \frac{L}{2}$ and do not follow a white-noise spectrum, the streamfunction perturbations will be spatially correlated in physical space. The Fourier expansion implies doubly periodic boundaries.

Since the physical processes mimicked by this streamfunction tendency forcing have finite correlation times, we introduce temporal correlations by evolving each spectral coefficient by a first-order autoregressive (AR1) process:

$$\psi'_{k,l}(t + \Delta t) = (1 - \alpha) \psi'_{k,l}(t) + g_{k,l} \sqrt{\alpha} \epsilon_{k,l}(t) \quad (6)$$

where $1 - \alpha$ is the linear autoregressive parameter, $g_{k,l}$ the wavenumber-dependent noise amplitude and $\epsilon_{k,l}$ a complex-valued Gaussian white-noise process with mean $\langle \epsilon_{k,l}(t) \rangle = 0$ and covariance $\langle \epsilon_{k,l}(s) \epsilon_{m,n}^*(t) \rangle = \sigma^2 \delta_{k,m} \delta_{l,n} \delta_{s,t}$. The $*$ denotes the complex conjugate. In addition, we assume $\alpha \in (0, 1]$ i.e., we exclude the cases of a non-fluctuating or decaying forcing. The variance and autocorrelation of the AR1 are well known quantities (e.g., Storch and Zwiers,

1999) and are given for the Markov process in (6) by:

$$\langle \psi'_{k,l}(t) \psi'^*_{k,l}(t) \rangle = \frac{g_{k,l}^2 \sigma^2 \alpha}{1 - (1 - \alpha)^2} = \frac{g_{k,l}^2 \sigma^2 \alpha}{\alpha(2 - \alpha)} = \frac{g_{k,l}^2 \sigma^2}{(2 - \alpha)} \quad (7)$$

where the autoregressive parameter $1 - \alpha$ is defined in terms of the (discrete) timestep Δt via

$$\alpha = \frac{\Delta t}{\tau}. \quad (8)$$

Furthermore we assume that the noise amplitudes follow the power-law

$$g_{k,l} = b \rho_{k,l}^\beta \quad (9)$$

with amplitude

$$b = \left(\frac{B_\psi \alpha}{2\pi^2 \sigma^2 \Gamma_\psi \Delta t} \right)^{\frac{1}{2}}, \quad \text{where} \quad \Gamma_\psi = \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^{2\beta+2}. \quad (10)$$

The radial wavenumber is $\rho_{k,l}$ is given by $\rho_{k,l} = \sqrt{k^2/X^2 + l^2/Y^2}$ and B_ψ denotes the backscatter rate as $B_\psi = \frac{\Delta E_{\text{kin}}}{\Delta t} = E'_{\text{kin}}$. And derived in Berner et al. (2009), this choice of b is such that at each timestep the total kinetic energy of the full flow is changed by:

$$E'_{\text{kin}} = \frac{2\pi^2}{\Delta t} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^2 \langle |\psi_{k,l}(t + \Delta t)|^2 - |\psi_{k,l}(t)|^2 \rangle \quad (11)$$

$$= \frac{2\pi^2}{\Delta t} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^2 \langle |\Delta \psi_{k,l}(t)|^2 \rangle \quad (12)$$

$$= 2\pi^2 \Delta t \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \left(\frac{2}{\alpha} - 1 \right) \rho_{k,l}^2 \langle |\psi'_{k,l}(t)|^2 \rangle \quad (13)$$

$$\stackrel{(7)}{=} \frac{2\pi^2 \sigma^2 \Delta t}{(2 - \alpha)} \frac{(2 - \alpha)}{\alpha} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^2 g_{k,l}^2 \quad (14)$$

$$\stackrel{(8)}{=} 2\pi^2 \sigma^2 \tau \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^2 g_{k,l}^2 \quad (15)$$

$$= 2\pi^2 \sigma^2 \tau \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^{2\beta+2} b^2 \quad (16)$$

$$= B_\psi. \quad (17)$$

We note that (16) implies (10) and that E'_{kin} is independent of Δt , $\frac{1}{K}$ and $\frac{1}{L}$, if each is small. Equation (13) is derived in appendix A (a copy of the appendix of Berner et al. (2009)) and states that the change of total kinetic energy is not strictly proportional to the variance of the forcing, $2\pi^2(\Delta t)^2 \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^2 \langle |\psi'_{k,l}(t)|^2 \rangle$, but is modified by the factor $(\frac{2}{\alpha} - 1)$. Berner et al. (2009) show that this modification is noise-induced and reflects the correlations between the total streamfunction $\psi(x, y, t)$ and the streamfunction forcing $\psi'(x, y, t)$ at time t due to their mutual dependence on the streamfunction forcing at the previous time $t - \Delta t$. If there are no such correlations (i.e., $\alpha = 1$ in the evolution equation (6)), this factor equals one and the change in total kinetic energy equals that of the variance of the forcing assuming that the forcing increments are instantaneously injected at each time step.

Secondly, we remark that the perturbation kinetic energy spectrum follows

$$E_{k,l}^{\text{kin}} \propto \rho_{k,l}^2 g_{k,l}^2 \quad (18)$$

$$\propto \rho_{k,l}^{2\beta+2}, \quad (19)$$

which states that a streamfunction forcing with power-law $\rho_{k,l}^\beta$ will result in a forcing kinetic-energy forcing of power-law $\rho_{k,l}^{2\beta+2}$.

An example of the spectral amplitudes for the streamfunction forcing is given in Fig. 1. The associated perturbation kinetic energy together with its power law behavior is displayed in Fig. 2. According to (13), the relation between the backscatter rate for streamfunction and the area A under the perturbation kinetic energy spectrum curve, $A = \rho_{k,l}^2 \langle |\psi'_{k,l}(t)|^2 \rangle$ are given by $B_\psi = 2\pi\Delta t \frac{(2-\alpha)}{\alpha} A$.

c. Derivation for potential temperature tendency perturbations

Let $\theta'(x, y, t)$ be a 2D potential temperature tendency forcing, $\theta'(x, y, t) = \frac{d\theta}{dt}$,

$$\theta'(x, y, t) = \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \theta'_{k,l}(t) e^{2\pi i(kx/X + ly/Y)} \quad (20)$$

Let the spectral coefficient describing the temperature tendency evolve as a first-order auto regressive (AR1) process:

$$\theta'_{k,l}(t + \Delta t) = (1 - \alpha)\theta'_{k,l}(t) + h_{k,l}\sqrt{\alpha}\epsilon_{k,l}(t). \quad (21)$$

with variance and auto correlation:

$$\langle \theta'_{k,l}(t)\theta'^*_{k,l}(t) \rangle = \frac{h_{k,l}^2\sigma^2}{2 - \alpha} \quad \text{and} \quad \frac{\langle \theta'_{k,l}(t + \Delta t)\theta'^*_{k,l}(t) \rangle}{\langle \theta'_{k,l}(t)\theta'^*_{k,l}(t) \rangle} = 1 - \alpha. \quad (22)$$

Furthermore assume that the noise amplitudes for temperature perturbations follow the power-law

$$h_{k,l} = f \rho_{k,l}^\gamma \quad (23)$$

with amplitude

$$f = \left(\frac{\theta_0\alpha B_\theta}{c_p\sigma^2\Gamma_\theta\Delta t} \right)^{\frac{1}{2}}, \quad \text{where} \quad \Gamma_\theta = \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \rho_{k,l}^{2\gamma}. \quad (24)$$

This choice of h is such that at each timestep Δt the potential energy of the flow is changed by $B_\theta = E'_{\text{pot}} = \frac{dE_{\text{pot}}}{dt}$.

$$E'_{\text{pot}} = \frac{c_p}{\theta_0\Delta t} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \langle |\theta_{k,l}(t + \Delta t)|^2 - |\theta_{k,l}(t)|^2 \rangle \quad (25)$$

$$= \frac{c_p}{\theta_0\Delta t} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \langle |\Delta\theta_{k,l}(t)|^2 \rangle \quad (26)$$

$$= \frac{c_p\Delta t}{\theta_0} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \left(\frac{2}{\alpha} - 1 \right) \langle |\theta'_{k,l}(t)|^2 \rangle \quad (27)$$

$$= \frac{c_p\sigma^2\Delta t}{\theta_0} \frac{(2 - \alpha)}{\alpha(2 - \alpha)} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} h_{k,l}^2 \quad (28)$$

$$= \frac{c_p\sigma^2\Delta t}{\theta_0\alpha} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} f^2 \rho_{k,l}^{2\gamma} \quad (29)$$

$$= \frac{c_p\sigma^2\Delta t}{\theta_0\alpha} \sum_{k=-K/2}^{K/2} \sum_{l=-L/2}^{L/2} \frac{\theta_0\alpha B_\theta}{c_p\sigma^2\Gamma_\theta\Delta t} \rho_{k,l}^{2\gamma} \quad (30)$$

$$= B_\theta. \quad (31)$$

In summary, a perturbation of the form (21) with noise amplitude (23) will generate temperature perturbations with the spectrum:

$$E_{\text{pot}}^{k,l} \propto h_{k,l}^2 \quad (32)$$

$$\propto f^2 \rho_{k,l}^{2\gamma}, \quad (33)$$

which states that a temperature forcing with power-law $\rho_{k,l}^\gamma$ will result in a forcing potential-energy spectrum with power-law $\rho_{k,l}^{2\gamma}$.

The spectral amplitude of the temperature forcing, its spectrum and power law behavior are displayed in Figs. 1,2. According to (27), the relation between the area $A = \langle |\theta'_{k,l}(t)|^2 \rangle$ under the potential energy spectrum curve and the backscatter rate for temperature are given by $B_\psi = \frac{c_p}{\theta_0} \Delta t \frac{(2-\alpha)}{\alpha} A$.

3. Version history and recommended namelist parameters

In the following we document the version history of SKEBS and list recommended namelist parameters for each WRF release. For versions WRFV3.3 and WRFV3.3.1, Tab. 1 lists selected namelist parameters and (currently) hard-coded parameter values and connects them to the derivations in section 2.

a. SKEBS in version WRF3.1.1 (Pre-release and Manuscript)

All results of Berner et al. 2011 were done in with a branch of WRFV3.1.1, i.e., before the official release of SKEBS in WRFV3.3.

- For WRFV3.1.1 and WRFV3.3 the formulation for the stochastic tendency perturbations follows that of the physics tendencies, which are coupled to mass μ (in the model $\mu = \text{MU} + \text{MUB}$). The value of the backscatter rates expresses the magnitude of mass-coupled tendencies, e.g.

$$\mu \frac{du_{\text{stoch}}}{dt} \propto \left(\frac{(\mu^2 B_\psi) \alpha}{2\pi^2 \sigma^2 \Gamma_\psi \Delta t} \right)^{\frac{1}{2}}, \quad (34)$$

Description	Symbol in derivation	Namelist name	Rec. (def.) value in WRFV3.3	Rec. (def.) value in WRFV3.3.1
Backscatter rate for streamfunction	B_ψ	TOT_BACKSCAT_PSI	115200 (115200)	1.0E-05 (1.7E-05)
Backscatter rate for temperature	B_θ	TOT_BACKSCAT_T	2.0 (2.0)	1.0E-06 (4.6E-05)
Decorrelation time	τ	TAU	$\sim 0.5\text{h}$	$\sim 0.5\text{h}$
Power law for streamfunction perturbations	β	REXPONENT_PSI	-1.83	-1.83
Power law for potential temperature perturbations	γ	REXPONENT_T	-1.83	-1.83

TABLE 1. Description and recommended values of selected namelist or hard-coded parameters

where the mass μ is of order $o(10^5)\text{Pa}$. In addition, the streamfunction tendency was erroneously coded as increment ($\Delta\psi$) rather than tendency ($\frac{d\psi}{dt}$), so that the value of TOT_BACKSCAT_PSI really stands for $\frac{\mu^2 B_\psi}{\Delta t^2} = \mu^2 \text{TOT_BACKSCAT_PSI}/\Delta t^2$ and is tuned to a value of 2.0 as reported in Berner et al. 2011.

- Analogously, the temperature tendency is assumed to be mass-coupled

$$\mu \frac{d\theta \text{stoch}}{dt} \propto \left(\frac{\alpha(\mu^2 B_\theta)}{2\pi^2 \sigma^2 \Gamma_\theta \Delta t} \right)^{\frac{1}{2}}, \quad (35)$$

and the mass-coupling is again folded-in for the value of the backscattered dissipation rate for temperature, resulting in a value of TOT_BACKSCAT_T = 2.0E-06.

- Noise variance: $\sigma^2 = 1/12$
- Model time step: $\Delta t = 240$
- Decorrelation time: $\tau = 1667\text{s} = 0.46\text{h}$

- This results in an autoregressive parameter of $1 - \alpha = 1 - \frac{\tau}{\Delta t} = 0.875$.

b. SKEBS in version WRF3.3 (Original Release)

- For the release version, the values of the backscatter rates still have the mass weighting folded in, but the streamfunction increment is now correctly coded as tendency. The results in backscatter rates of $\text{TOT_BACKSCAT_PSI} = 2.0\Delta t^2 = 2 \times 240^2 = 115200$ (bit-identical change) and $\text{TOT_BACKSCAT_T} = 2.0\text{E-}06$.
- The power law exponents are set to $\beta = \gamma = -1.83$, leading to a $-5/3$ -spectrum in perturbation kinetic energy and a $\gamma = -1.83^2$ -spectrum for perturbation potential energy as function of radial wavenumber ρ .
- All other parameters are the same as in WRFV3.1.1.

c. SKEBS in version WRF3.3.1

Due to user request the stochastic tendencies were decoupled from mass and the formulation for the potential temperature perturbations was changed. In hindsight these changes and in particular their combination was unfortunate, since they require a change of the default namelist parameters.

- Since the mass coupling factor is of order $o(10^5)\text{Pa}$, the decoupled stochastic tendencies for u and v are on average smaller by a factor 10^5 over the coupled tendencies. Since $\frac{du_{\text{stoch}}}{dt} \propto \sqrt{B_\psi}$, the backscatter rate for streamfunction was adjusted $\text{TOT_BACKSCAT_PSI} = \frac{115200}{(10^5)^2}$ to get perturbations of the same order as before. For simplicity (and to avoid suggesting that the rates were tuned to several decimal places) the value was then rounded to $\text{TOT_BACKSCAT_PSI} = 1.0\text{E-}05$. The default value in the registry is erroneously set to $1.7\text{E-}05$ which is too large for most applications.

- Rather than using the formulation (2) the temperature perturbations were set to

$$\frac{d\theta_{\text{stoch}}}{dt} \propto \frac{B_\theta}{c_p} \left(\frac{\alpha}{4\pi^2\sigma^2\Gamma_\theta} \right)^{\frac{1}{2}}. \quad (36)$$

This expression was chosen since it results in the correct units. However, the scaling with regard to Δt is incorrect and can result in too strong temperature forcings for timesteps of $< \Delta t = 100\text{s}$ or so¹.

- For timesteps in the range of 180s - 240s, perturbations of the same order as in WRFV3.3 are obtained, if the the value of the dissipation rate for temperature is set to TOT_BACKSCAT_T= 1.0E-06. This value is again rounded to several decimal places. The default value in the registry is erroneously set to 4.6E-05 and for most applications chosen too high.
- Note that this change made the potential temperature perturbations proportional to B_θ , while the streamfunction perturbations remain proportional to $\sqrt{B_\psi}$. So doubling TOT_BACKSCAT_T and TOT_BACKSCAT_PSI will lead to a relative increase of the temperature perturbations to the streamfunction perturbations.
- **“I ran my simulations with SKEBS in WRFV3.3.1. Do I need to redo my simulations.”** No, in my opinion not at all. This documentation just states why the user may get slightly different results in the future and how to interpret the results he has. None of the issues documented here invalidates the results obtained in WRFV3.3.1.

d. SKEBS in version WRFV3.4

No changes over WRFV3.3.1.

e. SKEBS in version WRFV3.5

No changes over WRFV3.3.1.

¹Thanks to Glenn Creighton, Air Force Weather Agency, for bringing this issue to our attention.

f. Known bugs to be corrected in future releases and recommendations

Below is a list of known bugs which will be corrected in future versions. Due to the modular nature of the SKEBS implementation, interested users can download and test a version of the module with the latest changes `dyn_em/module_stoch.F` available at http://www.cgd.ucar.edu/~berner/module_stoch.F. This module should work in any version of WRF, but a recompilation of the source code is necessary.

In addition this section contains a discussion of some of the (currently) hard-coded parameter values, which the scheme is sensitive to. Users might want to adjust these parameters depending on their application. To facilitate this, we intend to make these parameters namelist parameters in future releases. We do not intend to change the default values in the registry, since it would change the behavior of the scheme in regard to previous versions.

I) KNOWN BUGS

- Following (2) the temperature perturbations should be proportional to

$$\frac{d\theta_{\text{stoch}}}{dt} \propto \left(\frac{\theta_0 \alpha B_\theta}{c_p \sigma^2 \Gamma_\theta \Delta t} \right)^{\frac{1}{2}} \quad (37)$$

(see (24)). The formulation introduced in WRFV3.3.1. together with the recommended backscatter rate for temperature might lead to too large forcings, if a timestep of less than 100s is used (see Fig.3). The new formulation (37) should automatically adjust the forcing for any timestep.

- The maximum longitudinal and meridional perturbation wavenumbers were erroneously truncated to 40. This leads to a drop for larger wavenumbers (> 40) in the perturbation kinetic energy spectra (see Fig.3). Generally speaking, the perturbations at the largest scales will lead to the largest error growth, while perturbations at small scales tend to be damped (e.g., Tribbia and Baumhefner, 2004). Hence we do not expect a noticeable impact due to the erroneous truncation of wavenumbers > 40 .

II) DISCUSSION OF SELECTED (CURRENTLY) HARD-CODED PARAMETER VALUES

- If the slope of perturbation kinetic energy as function of radial wavenumber follows a power law behavior of $r^{2\beta+2}$, it will follow a power law of $k^{2\beta+2+1}$ in terms of **longitudinal** wavenumber k (see appendix B). E.g., should the perturbation kinetic energy spectrum have a power law behavior of $k^{-\frac{5}{3}}$ in terms of **longitudinal** wavenumber k , the parameter β should be set to $\beta = (-\frac{5}{3} - 3)/2 = -2.33$ (19).
- The decorrelation time of $\tau \sim 0.5h$ seems rather short. A recommended values is $\tau = 3h = 10800s$. However, since the choice of τ will affect the injected energy (13,27) the backscatter rates might need to be re-tuned.
- The autoregressive process should be spun up (in `start_em.F`) before the first update.

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APPENDIX A. Derivation of backscattered energy

This appendix is copied from Berner et al.(2009) and derives the global kinetic energy that streamfunction perturbations of the form (6) inject into the resolved flow. Note that this derivation was done for spherical harmonics. For definitions and details, we refer to Berner et al.(2009).

The injected energy per unit mass, $\Delta E'$, is the difference between the total kinetic energy per unit mass at time $t + \Delta t$ and t :

$$\Delta E' = \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \frac{n(n+1)}{a^2} \langle |\psi_n^m(t + \Delta t)|^2 - |\psi_n^m(t)|^2 \rangle. \quad (1)$$

The change in total kinetic energy is expressed in terms of spherical harmonics and takes the spherical symmetry into account (e.g., Koshyk and Hamilton, 2001). The streamfunction at time $t + \Delta t$ is given by

$$\psi_n^m(t + \Delta t) = \psi_n^m(t) + S_n^m(t)\Delta t + \psi_n'^m(t), \quad (2)$$

where S_n^m is the spectral source term due to advection, diffusion and physical parameterizations. Using (2) we arrive at:

$$\Delta E' = \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \frac{n(n+1)}{a^2} \langle |\psi_n^m(t) + S_n^m(t)\Delta t + \psi_n'^m(t)|^2 - |\psi_n^m(t)|^2 \rangle \quad (3)$$

$$= \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \frac{n(n+1)}{a^2} \langle 2|\psi_n^m(t)\psi_n'^m(t)| + |\psi_n'^m(t)|^2 \rangle \quad (4)$$

$$+ \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \frac{n(n+1)}{a^2} \langle 2|(\psi_n^m(t) + \psi_n'^m(t))S_n^m(t)\Delta t|^2 + |S_n^m(t)\Delta t|^2 \rangle \quad (5)$$

While the first term describes the energy injection due to interactions between the resolved flow and the forcing, the second is directly associated with the energy injected by the streamfunction perturbations. The third and forth term contain the spectral source term S_n^m and its interaction with the resolved flow. Since the source term is not correlated with the streamfunction perturbations, we can set $\langle |\psi_n'^m(t)S_n^m(t)\Delta t|^2 \rangle = 0$.

Let's have a closer look at the interaction term $\langle 2|\psi_n^m(t)\psi_n'^m(t)|\rangle$. To calculate this term we start from the propagation equation (2), multiply it by $\psi_n'^m(t + \Delta t)$ and take the ensemble average:

$$\langle \psi_n^m(t + \Delta t)\psi_n'^m(t + \Delta t) \rangle = \langle \psi_n^m(t)\psi_n'^m(t + \Delta t) \rangle + \langle S_n^m(t)\psi_n'^m(t + \Delta t) \rangle \Delta t + \langle \psi_n'^m(t)\psi_n'^m(t + \Delta t) \rangle . \quad (6)$$

Using (6) to substitute $\psi_n'^m(t + \Delta t)$ and noting that $\langle \psi_n^m(t)\epsilon(t) \rangle = 0$, $\langle \psi_n'^m(t)\epsilon(t) \rangle = 0$ and $\langle |\psi_n'^m(t)S_n^m(t)\Delta t|^2 \rangle = 0$, we arrive at:

$$\langle \psi_n^m(t + \Delta t)\psi_n'^m(t + \Delta t) \rangle = (1 - \alpha) [\langle \psi_n^m(t)\psi_n'^m(t) \rangle + \langle \psi_n'^m(t)\psi_n'^m(t) \rangle] . \quad (7)$$

Using the stationarity condition $\langle \psi_n^m(t + \Delta t)\psi_n'^m(t + \Delta t) \rangle = \langle \psi_n'^m(t)\psi_n'^m(t) \rangle$ yields:

$$\langle \psi_n^m(t)\psi_n'^m(t) \rangle = \left(\frac{1}{\alpha} - 1\right) \langle \psi_n'^m(t)\psi_n'^m(t) \rangle . \quad (8)$$

Note, that for $\alpha = 0$, $\langle \psi_n'^m(t)\psi_n'^m(t) \rangle = \text{const}$ and we cannot derive (8) from (7), which is why we exclude this case. Inserting (8) into (5) the total injected kinetic energy is derived to:

$$\Delta E' = \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \left(\frac{2}{\alpha} - 1\right) \frac{n(n+1)}{a^2} \langle |\psi_n'^m(t)|^2 \rangle \quad (9)$$

$$+ \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \frac{n(n+1)}{a^2} \langle 2|\psi_n^m(t)S_n^m(t)\Delta t|^2 + |S_n^m\Delta t|^2 \rangle . \quad (10)$$

To make the problem tractable we neglect all terms with the source and sink term $S_n^m(t)$. There is no strict justification for making this assumption, but since the stochastic kinetic backscatter algorithm makes many assumptions and simplifications, we are not overly concerned about neglecting these terms. With this assumptions the injected kinetic energy is given by

$$\Delta E' = \frac{1}{4\pi} \sum_{n=0}^N \sum_{m=-n}^n \left(\frac{2}{\alpha} - 1\right) \frac{n(n+1)}{a^2} \langle |\psi_n'^m(t)|^2 \rangle . \quad (11)$$

If the noise process had no temporal memory, $\alpha = 1$, and forcing increments were injected instantaneously at each time step, the injected energy would equal the kinetic energy of the

streamfunction perturbations. However, since the noise process has temporal correlations, the streamfunction and streamfunction perturbations are correlated and these correlations lead to an increase in the injected energy (5).

With (22) and noting that $\sum_{n=1}^N \sum_{m=-n}^n n(n+1) = \sum_{n=1}^N n(n+1)(2n+1)$, we arrive at the following analytic expression for the perturbation energy:

$$\Delta E' = \frac{\sigma_z}{4\pi a^2} \frac{1}{\alpha} \sum_{n=1}^N n(n+1)(2n+1)g_n^2 \quad (12)$$

Assuming furthermore that the noise amplitudes g_n follow the power-law $g_n = bn^p$ and that the globally-averaged kinetic energy increment $\Delta E'$ is given and fixed, we are solving (12) for the amplitude function b ,

$$b = \left(\frac{4\pi a^2}{\sigma_z \Gamma} \alpha \Delta E' \right)^{\frac{1}{2}}, \quad (13)$$

where $\Gamma = \sum_{n=1}^N n(n+1)(2n+1)n^{2p}$.

a. *Radial wavenumber*

Here, we derive how a power law as function of radial wavenumber (as defined in SKEBS) translates into a power law as function of longitudinal wavenumber. Latter formulation tends to be more available from observations, e.g. Nastrom and Gage, 1985.

Let x and y be the zonal and meridional coordinates on a 2D Cartesian plane. The kinetic energy is defined as:

$$T = \frac{1}{2XY} \int_{x=-X/2}^{X/2} \int_{y=-Y/2}^{Y/2} [u(x,y)^2 + v(x,y)^2] dx dy \quad (14)$$

$$= \frac{1}{XY} \int_{x=-X/2}^{X/2} \int_{y=-Y/2}^{Y/2} e(x,y) dx dy, \quad (15)$$

$$\text{with } \begin{cases} e(x,y) = \frac{1}{2} [u(x,y)^2 + v(x,y)^2], \text{ for } \frac{-X}{2} < x < \frac{X}{2} \text{ and } \frac{-Y}{2} < y < \frac{Y}{2} \\ e(x,y) = 0, \text{ for } |x| > \frac{X}{2} \text{ or } |y| > \frac{Y}{2} \end{cases} \quad (16)$$

Assuming homogeneity, turbulence theory defines the kinetic energy as integral of the spectrum as function of the radial wavenumber $r = (k^2 + l^2)^{\frac{1}{2}}$:

$$T = \int_r E(r) dr. \quad (17)$$

In the following we solve for $E(r)$, so that we get the relationship between the kinetic-energy spectrum in Cartesian and polar coordinates.

Expressing (16) in polar coordinates, and comparing to (17) yields:

$$T = \int_{x=-X/2}^{X/2} \int_{y=-Y/2}^{Y/2} e(x,y) dx dy \quad (18)$$

$$= \int_{r=0}^{\sqrt{X^2+Y^2}/4} \underbrace{\int_{\phi=-\pi}^{\pi} e(r \cos \phi, r \sin \phi) r d\phi dr}_{E(r)} \quad (19)$$

Solving for $E(r)$ we get:

$$E(r) = \int_{\phi=-\pi}^{\pi} e(r \cos \phi, r \sin \phi) r d\phi \quad (20)$$

$$= \int_{x=-X/2}^{X/2} \int_{y=-Y/2}^{Y/2} e(x,y) \frac{xdy - ydx}{r}. \quad (21)$$

From (21) we see that if the spectrum of $E(r)$ follows a power law of $E(r) \sim r^p$ as function of radial wavenumber r then it follows a power law with slope $E(k) \sim k^{p-1}$ as function of longitudinal wavenumber k .

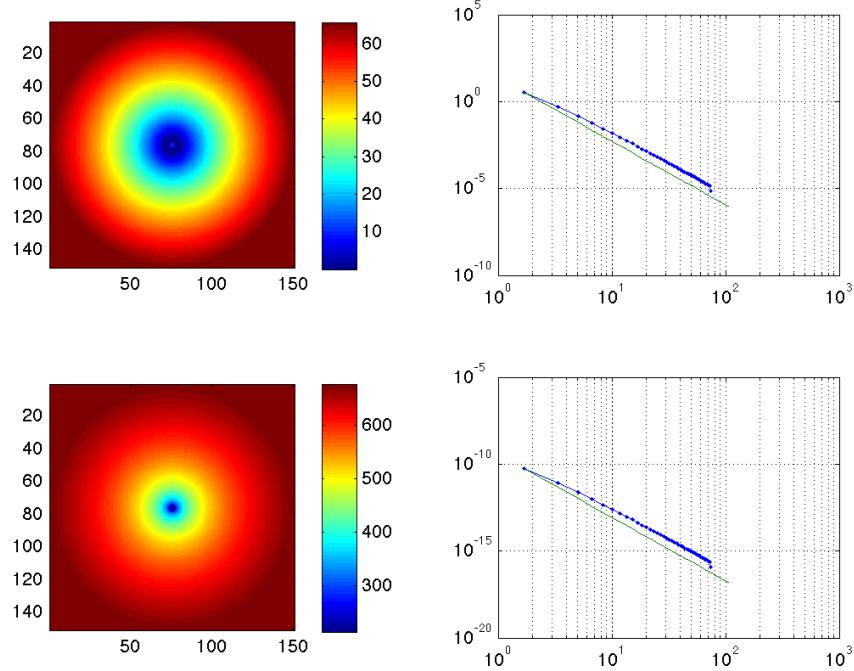


FIG. 1. Spectral amplitudes of streamfunction (upper) and potential temperature (lower) forcing for domain with extension 5436 km. Shown are amplitudes in 2D wavenumber space (left) and as function of radial wavenumber (right). Black lines denote power laws of $r^{2\beta}$, where r is the radial wavenumber.

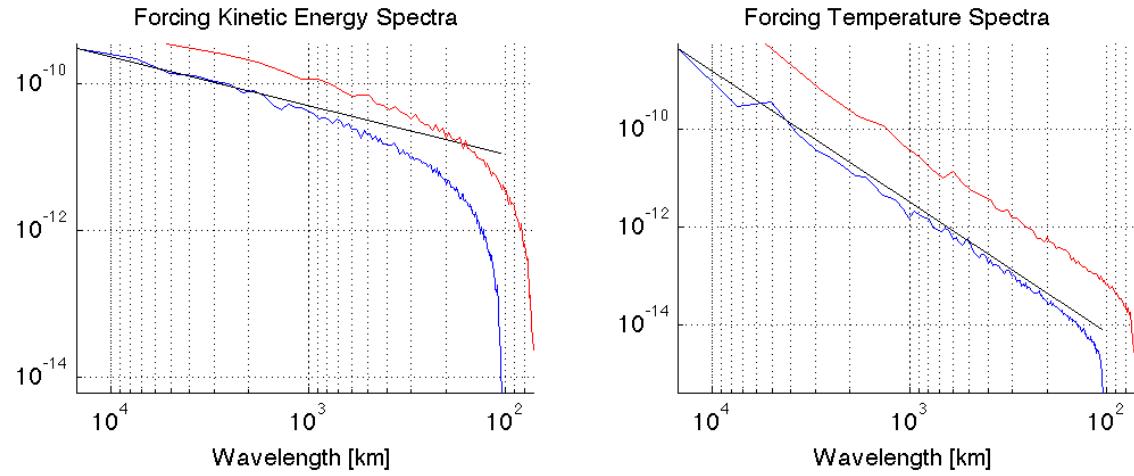


FIG. 2. Kinetic energy and temperature forcing spectra for two different domains of extension (blue: 15132 km (at equator) and red: 5436 km) and horizontal resolutions (50km, 36km). Black lines denote power laws of $k^{(2\beta+2)+1}$ (left) and $k^{(2\gamma+1)}$ (right), where k is the longitudinal wavenumber (see Eqs.(19,33) and appendix).

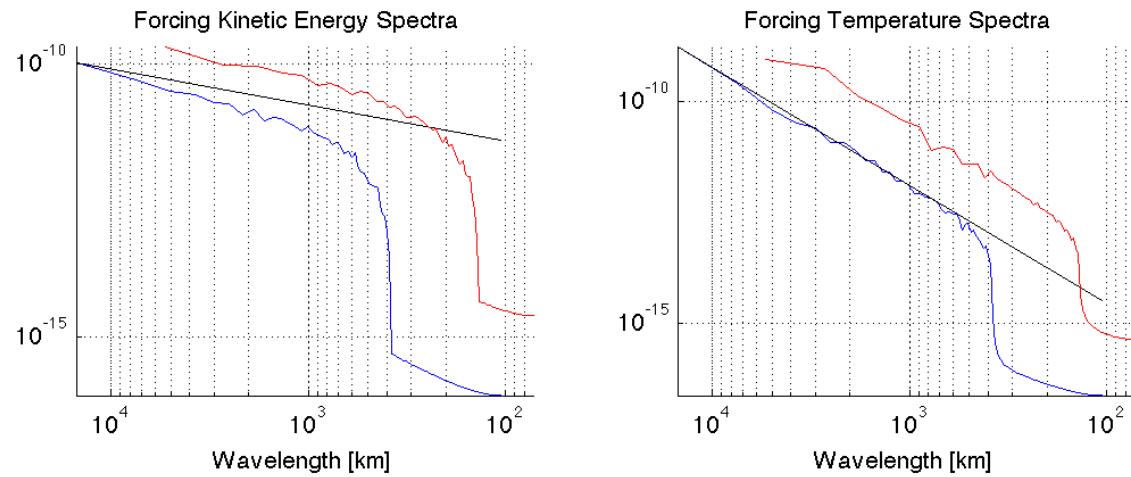


FIG. 3. Same as Fig. 2 but for SKEBS release in version WRFV3.3.1 (and earlier). Wavenumber of $k > 40$ were not forced leading to the steep drop-off at large wavenumbers, more so for u and v since they are computed as streamfunction derivatives.

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