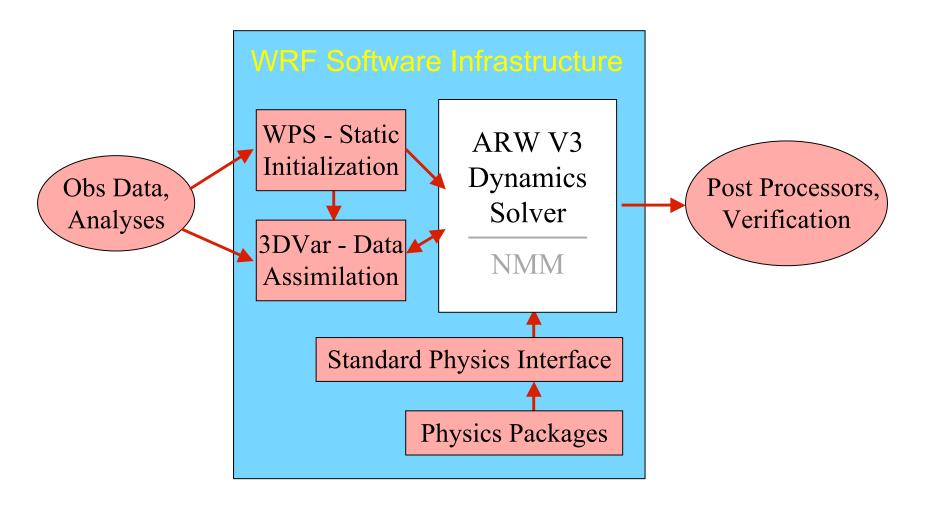
The Advanced Research WRF (ARW) Dynamics Solver

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WRF ARW Tech Note

A Description of the Advanced Research WRF Version 3 http://www.mmm.ucar.edu/wrf/users/pub-doc.html

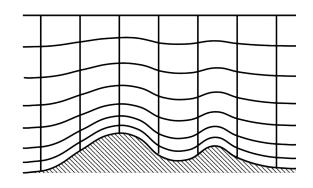
ARW Dynamical Solver

- Terrain representation
- Vertical coordinate
- Equations / variables
- Time integration scheme
- Grid staggering
- Advection scheme
- Time step parameters
- Filters
- Boundary conditions
- Nesting
- Map projections

ARW, Terrain Representation

Lower boundary condition for the geopotential ($\phi = gz$) specifies the terrain elevation, and specifying the lowest coordinate surface to be the terrain results in a terrain-following coordinate.

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + \omega \frac{\partial \phi}{\partial \eta} = gw$$



Vertical coordinate:

hydrostatic pressure
$$\boldsymbol{\pi}$$
 $\eta = \frac{(\pi_d - \pi_{dt})}{\mu_d}, \quad \mu_d = \pi_{ds} - \pi_{dt}$

Flux-Form Equations in ARW

Terrain-following hydrostatic pressure coordinate:

hydrostatic pressure π

$$\eta = \frac{(\pi_d - \pi_{dt})}{\mu_d}, \quad \mu_d = \pi_{ds} - \pi_{dt}, \quad \mu_d(x, y)\Delta \eta = \Delta \pi_d = -g\rho_d \Delta z$$

Conserved state variables:

$$\mu_d$$
, $U = \mu_d u$, $V = \mu_d v$, $W = \mu_d w$, $\Theta = \mu_d \theta$

Non-conserved state variable: $(\phi = gz)$

2D Flux-Form Moist Equations in ARW

Moist Equations:

$$\frac{\partial U}{\partial t} + \alpha \mu_d \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g \left(\mu_d - \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial \Omega\theta}{\partial \eta} = \mu Q$$

$$\frac{\partial \Phi}{\partial t} = gw$$

$$\frac{\partial (\mu_d q_{v,l})}{\partial t} + \frac{\partial (U q_{v,l})}{\partial x} + \frac{\partial (\Omega q_{v,l})}{\partial \eta} = \mu Q_{v,l}$$

Diagnostic relations:

$$\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, \quad p = \left(\frac{R\Theta}{p_o \mu_d \alpha_v}\right)^{\gamma}$$

Time Integration in ARW

3rd Order Runge-Kutta time integration

advance
$$\phi^t \rightarrow \phi^{t+\Delta t}$$

$$\phi^* = \phi^t + \frac{\Delta t}{3} R(\phi^t)$$

$$\phi^{**} = \phi^t + \frac{\Delta t}{2} R(\phi^*)$$

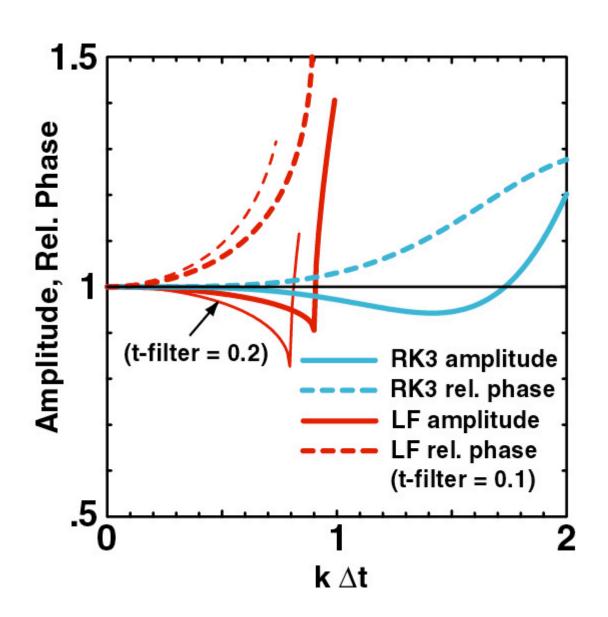
$$\phi^{t+\Delta t} = \phi^t + \Delta t R(\phi^{**})$$

Amplification factor
$$\phi_t = i k \phi$$
; $\phi^{n+1} = A \phi^n$; $|A| = 1 - \frac{(k\Delta t)^4}{24}$

Phase and amplitude errors for LF, RK3

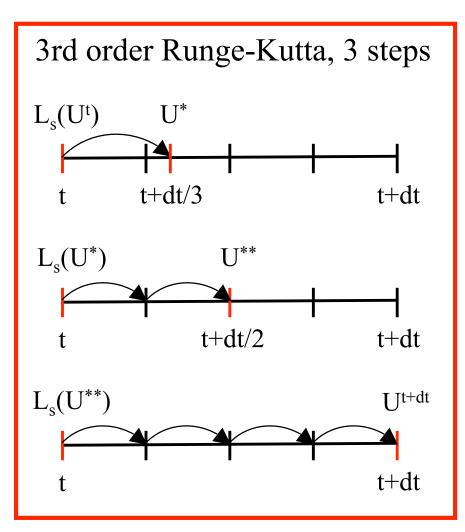
Oscillation equation analysis

$$\phi_t = ik\phi$$



Time-Split Runge-Kutta Integration Scheme

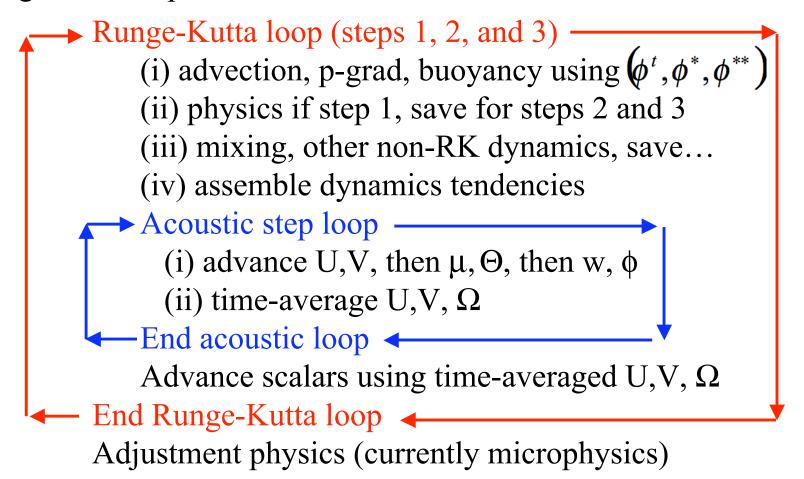
$$U_{t} = L_{fast}(U) + L_{slow}(U)$$



- RK3 is 3rd order accurate for linear eqns, 2nd order accurate for nonlinear eqns.
- Stable for centered and upwind advection schemes.
- Stable for Courant number Udt/dx < 1.73
- Three $L_{slow}(U)$ evaluations per timestep.

WRF ARW Model Integration Procedure

Begin time step



End time step

Flux-Form Perturbation Equations

Introduce the
$$\phi = \overline{\phi}(\overline{z}) + \phi', \ \mu = \overline{\mu}(\overline{z}) + \mu';$$
 perturbation variables: $p = \overline{p}(\overline{z}) + p', \ \alpha = \overline{\alpha}(\overline{z}) + \alpha'$

Note –
$$\phi = \overline{\phi}(\overline{z}) = \overline{\phi}(x, y, \eta),$$

likewise $\overline{p}(x, y, \eta), \overline{\alpha}(x, y, \eta)$

Reduces horizontal pressure-gradient errors.

For small time steps, recast variables as perturbations from time t

$$U' = U'^{t} + U'', \ V' = V'^{t} + V'', \ W' = W'^{t} + W'',$$

$$\Theta' = \Theta'^{t} + \Theta'', \ \mu' = \mu'^{t} + \mu'', \ \phi' = \phi'^{t} + \phi'';$$

$$p' = p'^{t} + p'', \ \alpha' = \alpha'^{t} + \alpha''$$

Allows vertical pressure gradient to be expressed in terms of ϕ ".

Small Time Step Integration of Acoustic/Gravity Wave Terms

(Without expanding variables into perturbation form)

$$U^{\tau+\Delta\tau} \qquad \frac{\partial U}{\partial t} + \left(\mu_{d}\alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_{d}} \frac{\partial p}{\partial \eta} \frac{\partial \phi}{\partial x}\right)^{\tau} = R_{U}^{t}$$

$$\mu_{d}^{\tau+\Delta\tau} \qquad \Omega^{\tau} \qquad \frac{\partial \mu_{d}}{\partial t} + \frac{\partial U}{\partial x}^{\tau+\Delta\tau} + \frac{\partial \Omega}{\partial \eta}^{\tau+\Delta\tau} = 0$$

$$\Theta^{\tau+\Delta\tau} \qquad \frac{\partial \Theta}{\partial t} + \left(\frac{\partial U\theta^{t}}{\partial x} + \frac{\partial \Omega\theta^{t}}{\partial \eta}\right)^{\tau+\Delta\tau} = R_{\Theta}^{t}$$

$$W^{\tau+\Delta\tau} \qquad \frac{\partial W}{\partial t} + g\left(\mu_{d} - \frac{\alpha}{\alpha_{d}} \frac{\partial p}{\partial \eta}\right)^{\tau} = R_{W}^{t}$$

$$\phi^{\tau+\Delta\tau} \qquad \mu_{d}^{t} \frac{\partial \phi}{\partial t} + U^{\tau+\Delta\tau} \frac{\partial \phi^{t}}{\partial x} + \Omega^{\tau+\Delta\tau} \frac{\partial \phi^{t}}{\partial \eta} - g\overline{W}^{\tau} = R_{\phi}^{t}$$

- Forward-backward differencing on U, Θ , and μ equations
- Vertically implicit differencing on W and ϕ equations

Hydrostatic Option

Instead of solving vertically implicit equations for W and ϕ

Integrate the hydrostatic equation to obtain $p(\pi)$:

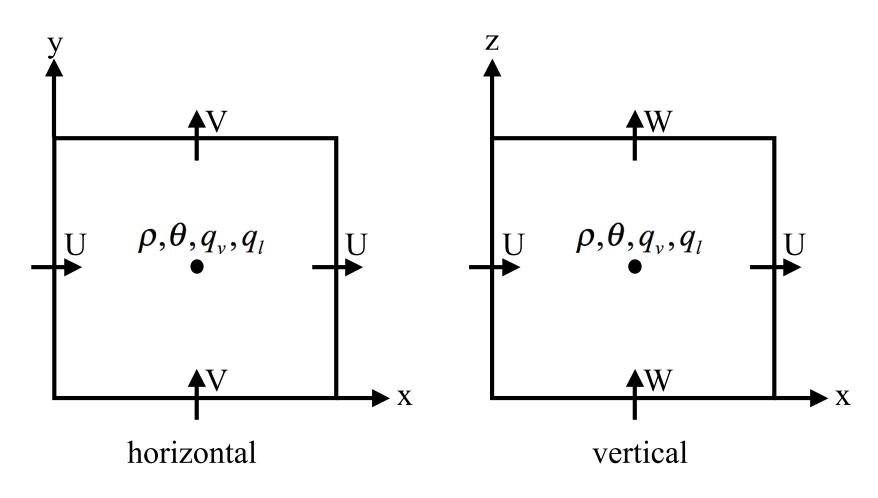
$$\frac{\partial p}{\partial \eta} = \left(\frac{\alpha_d}{\alpha}\right)^t \mu$$

Recover
$$\alpha$$
 and ϕ from: $p = p_0 \left(\frac{R\theta}{p_0 \alpha_v}\right)^{\gamma}$, and $\frac{\partial \phi}{\partial \eta} = -\mu_d \alpha_d$

W is no longer required during the integration.

ARW model, grid staggering

C-grid staggering



Advection in the ARW Model

2nd, 3rd, 4th, 5th and 6th order centered and upwind-biased schemes are available in the ARW model.

Example: 5th order scheme

$$\frac{\partial (U\phi)}{\partial x} = \frac{1}{\Delta x} \left(F_{i+\frac{1}{2}}(U\phi) - F_{i-\frac{1}{2}}(U\phi) \right)$$

where

$$F_{i-\frac{1}{2}}(U\phi) = U_{i-\frac{1}{2}} \left\{ \frac{37}{60} (\phi_{i} + \phi_{i-1}) - \frac{2}{15} (\phi_{i+1} + \phi_{i-2}) + \frac{1}{60} (\phi_{i+2} + \phi_{i-3}) \right\}$$

$$-sign(1,U) \frac{1}{60} \left\{ (\phi_{i+2} - \phi_{i-3}) - 5(\phi_{i+1} - \phi_{i-2}) + 10(\phi_{i} - \phi_{i-1}) \right\}$$

Advection in the ARW Model

For constant U, the 5th order flux divergence tendency becomes

$$\Delta t \frac{\delta \left(U\phi\right)}{\Delta x} \bigg|_{5th} = \Delta t \frac{\delta \left(U\phi\right)}{\Delta x} \bigg|_{6th}$$

$$- \left| \frac{U\Delta t}{\Delta x} \right| \frac{1}{60} \left(-\phi_{i-3} + 6\phi_{i-2} - 15\phi_{i-1} + 20\phi_{i} - 15\phi_{i+1} + 6\phi_{i+2} - \phi_{i+3}\right)$$

$$\frac{Cr}{60} \frac{\partial^{6}\phi}{\partial x^{6}} + H.O.T$$

The odd-ordered flux divergence schemes are equivalent to the next higher ordered (even) flux-divergence scheme plus a dissipation term of the higher even order with a coefficient proportional to the Courant number.

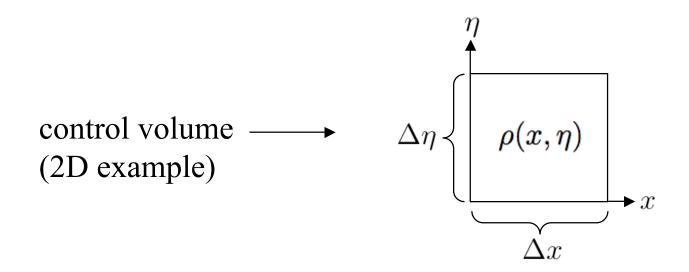
Maximum Courant Number for Advection

$$C_a = U\Delta t/\Delta x$$

Time Integration Scheme	Advection Scheme				
	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
Leapfrog (α=0.1)	0.91	U	0.66	U	0.57
RK2	U	0.90	U	0.39	U
RK3	1.73	1.63	1.26	1.43	1.09

U = unstable

(Wicker & Skamarock, 2002)



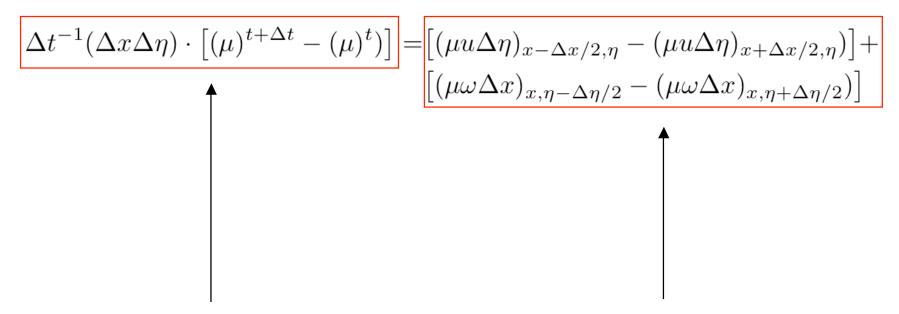
Mass in a control volume is proportional to

$$(\Delta x \Delta \eta)(\mu)^t$$

since
$$\mu(x)\Delta\eta = \Delta\pi = -g\rho\Delta z$$

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$ 2D example

Mass conservation equation



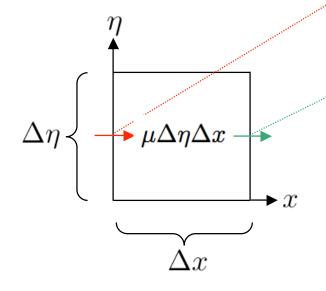
Change in mass over a time step

mass fluxes through control volume faces

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Mass conservation equation

$$\Delta t^{-1} (\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t \right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - \left[(\mu u \Delta \eta)_{x+\Delta x/2,\eta} \right] \right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2} \right]$$



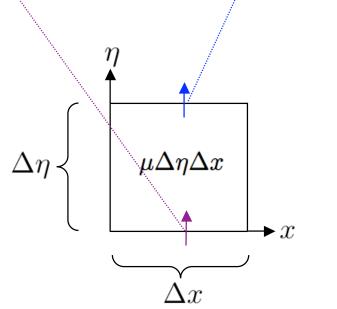
Horizontal fluxes through the vertical control-volume faces

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

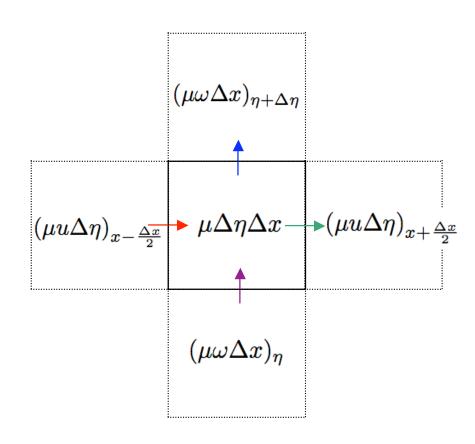
Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t \right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta} \right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2} \right]$$

Vertical fluxes through the horizontal control-volume faces



The same mass fluxes are used for neighboring grid cells - hence mass is conserved locally and globally.



Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$ Scalar mass $(\Delta x \Delta \eta)(\mu \phi)^t$

Mass conservation equation:

$$\frac{\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t \right]}{ \uparrow} = \frac{\left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta}) \right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2} \right] }{\left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2} \right] }$$

change in mass over a time step

mass fluxes through control volume faces

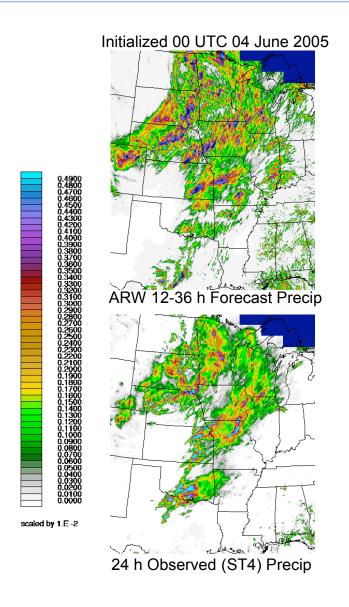
Scalar mass conservation equation:

$$\Delta t^{-1} (\Delta x \Delta \eta) \cdot \left[(\mu \phi)^{t+\Delta t} - (\mu \phi)^t \right] = \left[(\mu u \phi \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \phi \Delta \eta)_{x+\Delta x/2,\eta} \right] + \left[(\mu \omega \phi \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \phi \Delta x)_{x,\eta+\Delta \eta/2} \right]$$

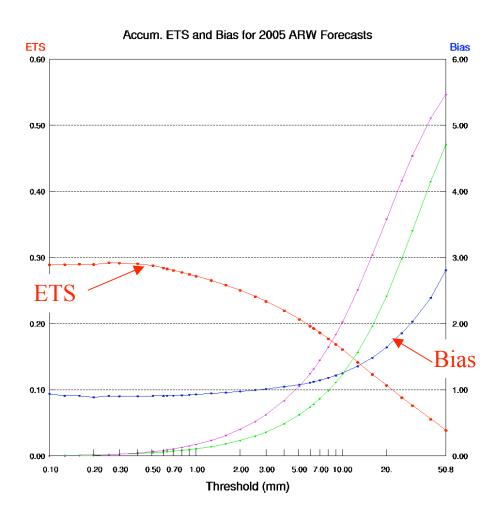
change in tracer mass over a time step

tracer mass fluxes through control volume faces

Moisture Transport in ARW: High Precipitation Bias

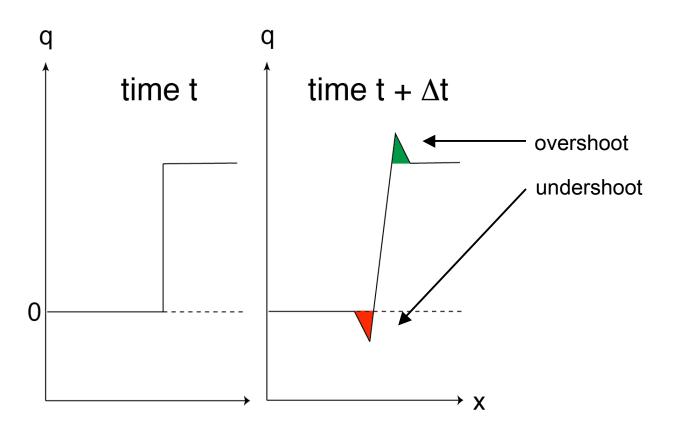


2005 ARW 4 km Forecasts:



Moisture Transport in ARW

1D advection



ARW scheme is conservative, but not positive definite nor monotonic. Removal of negative q results in spurious source of q .

Positive-Definite/Monotonic Flux Renormalization

Scalar update, last RK3 step

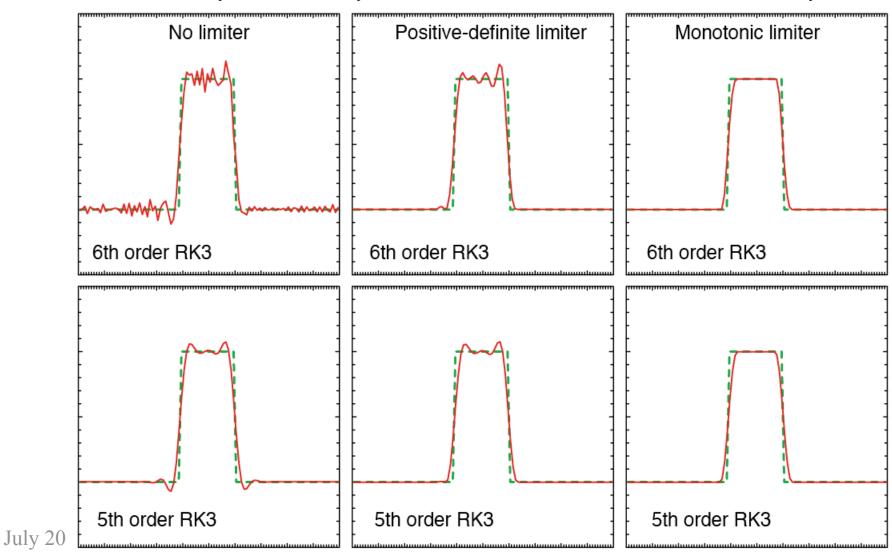
$$(\mu\phi)^{t+\Delta t} = (\mu\phi)^t - \Delta t \sum_{i=1}^n \delta_{x_i}[f_i]$$
 (1)

- (1) Decompose flux: $f_i = f_i^{upwind} + f_i^c$
- (2) Renormalize high-order correction fluxes f_i^c such that solution is positive definite or monotonic: $f_i^c = R(f_i^c)$
- (3) Update scalar eqn. (1) using $f_i = f_i^{upwind} + R(f_i^c)$

Skamarock, MWR 2006, 2241-2250

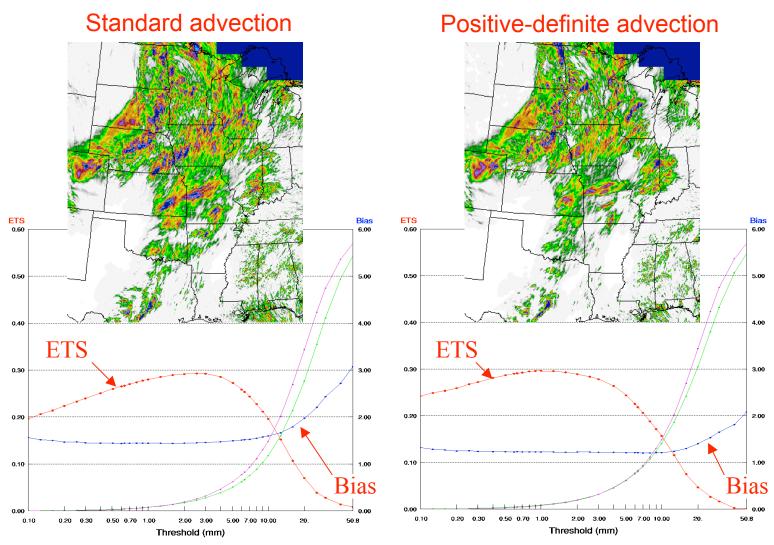
PD/Monotonic Limiters in ARW - 1D Example Top-Hat Advection

1D Top-hat transport Cr = 0.5, 1 revolution, 200 steps



Moisture Transport in ARW: 24 h ETS and BIAS





ARW Model: Dynamics Parameters

3rd order Runge-Kutta time step

Courant number limited, 1D:
$$C_r = \frac{U\Delta t}{\Delta x} < 1.43$$

Generally stable using a timestep approximately twice as large as used in a leapfrog model.

Acoustic time step

2D horizontal Courant number limited:
$$C_r = \frac{C_s \Delta \tau}{\Delta h} < \frac{1}{\sqrt{2}}$$

 $\Delta \tau_{sound} = \Delta t_{RK} / \text{(number of acoustic steps)}$

Guidelines for time step

 Δt in seconds should be about $6*\Delta x$ (grid size in kilometers). Larger Δt can be used in smaller-scale dry situations, but $time_step_sound$ (default = 4) should increase proportionately if larger Δt is used.

ARW Filters: Divergence Damping

Purpose: filter acoustic modes (3-D divergence, $D = \nabla \cdot \rho \mathbf{V}$)

$$\left\{ \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla p + \dots = \gamma_d' \nabla D \right\}$$

$$\nabla \cdot \left\{ \quad \right\} \quad \to \quad \frac{\partial D}{\partial t} + \nabla^2 p + \dots = \gamma_d' \nabla^2 D$$

From the pressure equation: $p_t \simeq c^2 D$

$$\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla [p_{\tau} + \gamma_d (p^{\tau} - p^{\tau - \Delta \tau})] + \dots = 0$$

 $\gamma_d = 0.1$ recommended (default)

(Illustrated in height coordinates for simplicity)

ARW Filters: Vertically Implicit Off-Centered Acoustic Step

Purpose: damp vertically-propagating acoustic modes

$$\frac{\partial W}{\partial t} + g \overline{\left(\mu_d - \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta}\right)^{\tau}} = \dots$$

$$\frac{\partial \phi}{\partial t} - \frac{g}{\mu_d^t} \overline{W}^{\tau} = \dots$$

$$\overline{()^{\tau}} = \frac{1 + \beta}{2} \overline{()^{\tau + \Delta \tau}} + \frac{1 - \beta}{2} \overline{()^{\tau}}$$

Slightly forward centering the vertical pressure gradient damps 3-D divergence as demonstrated for the divergence damper

 β = 0.1 recommended (default)

ARW Filters: External Mode Filter

Purpose: filter the external mode

Vertically integrated horizontal divergence, $D_h = \int_1^0 (\nabla_{\eta} \cdot \mu \mathbf{V}_h) d\eta$

$$\left\{ \frac{\partial \mu \mathbf{V}_h}{\partial t} + \ldots = -\gamma_e \nabla_{\eta} D_h \right\}$$

$$\int_{1}^{0} \nabla_{\eta} \cdot \left\{ \right\} d\eta \quad \to \quad \frac{\partial D_{h}}{\partial t} + \ldots = \gamma_{e} \nabla^{2} D_{h}$$

Hydrostatic continuity equation: $\frac{\partial \mu}{\partial t} = -\nabla_{\eta} \cdot \mu \mathbf{V}_h - \frac{\partial \mu \dot{\eta}}{\partial \eta} = D_h$

$$\frac{\partial \mu \mathbf{V}_h}{\partial \tau} + \dots = -\gamma_e \frac{\Delta x^2}{\Delta \tau^2} \nabla_{\eta} (\mu^{\tau} - \mu^{\tau - \Delta \tau})$$

 $\gamma_e = 0.01$ recommended (default)

(Primarily for real-data applications)

ARW Filters: Vertical Velocity Damping

Purpose: damp anomalously-large vertical velocities

(usually associated with anomalous physics tendencies)

Additional term:

$$\partial_t W = \dots - \mu_d \operatorname{sign}(W) \gamma_w (Cr - Cr_\beta)$$

$$Cr = \left| \frac{\Omega dt}{\mu d\eta} \right|$$

$$Cr_{\beta}$$
=1.0 typical value (default)
 γ_{w} = 0.3 m/s² recommended (default)

ARW Filters: 2nd-Order Horizontal Mixing, Horizontal-Deformation-Based K_h

Purpose: mixing on horizontal coordinate surfaces (real-data applications)

$$K_h = C_s^2 l^2 \left[0.25(D_{11} - D_{22})^2 + \overline{D_{12}^2}^{xy} \right]^{\frac{1}{2}}$$

where
$$l = (\Delta x \Delta y)^{1/2}$$

$$D_{11} = 2 m^{2} [\partial_{x}(m^{-1}u) - z_{x} \partial_{z}(m^{-1}u)]$$

$$D_{22} = 2 m^{2} [\partial_{y}(m^{-1}v) - z_{y} \partial_{z}(m^{-1}v)]$$

$$D_{12} = m^{2} [\partial_{y}(m^{-1}u) - z_{y} \partial_{z}(m^{-1}u) + \partial_{x}(m^{-1}v) - z_{x} \partial_{z}(m^{-1}v)]$$

 $C_s = 0.25$ (Smagorinsky coefficient, default value)

Implicit Rayleigh w Damping Layer for Split-Explicit Nonhydrostatic NWP Models

Modification to small time step:

 Step horizontal momentum, continuity, and potential temperature equations to new time level:

$$U^{\tau + \Delta \tau} \qquad \mu^{\tau + \Delta \tau}$$

$$\Omega^{\tau + \Delta \tau} \qquad \Theta^{\tau + \Delta \tau}$$

 Step vertical momentum and geopotential equations (implicit in the vertical):

$$W^{*\tau+\Delta\tau}$$
 $\phi^{*\tau+\Delta\tau}$

 Apply implicit Rayleigh damping on W as an adjustment step:

$$W^{\tau + \Delta \tau} = W^{*\tau + \Delta \tau} - \Delta \tau R_w(\eta) W^{\tau + \Delta \tau}$$

 Update final value of geopotential at new time level:

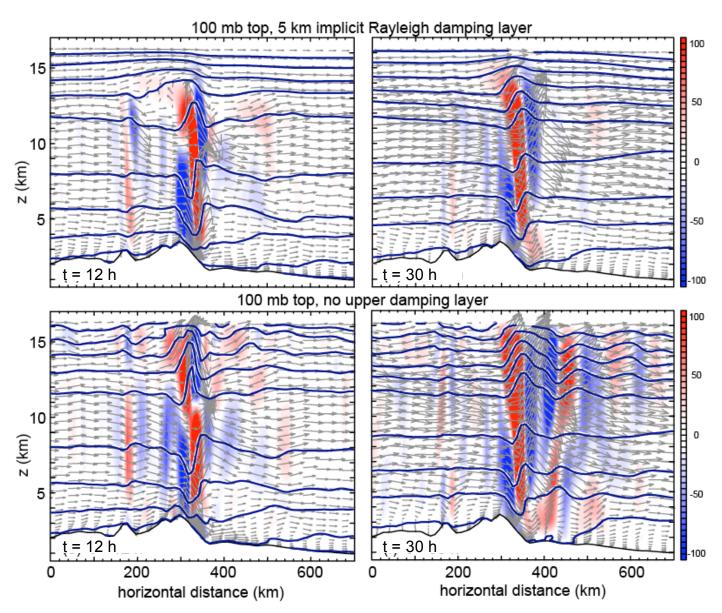
$$\phi^{\tau + \Delta \tau}$$

$$R_w(\eta) = \left\{ \begin{array}{ll} \gamma_r \sin^2\left[\frac{\pi}{2}\left(1 - \frac{z_{top} - z}{z_d}\right)\right] & \text{for } z \geq (z_{top} - z_d); \\ 0 & \text{otherwise,} \end{array} \right. \begin{array}{l} \tau(z) \text{ - damping rate (t$^-$1$)} \\ z_d \text{ - depth of the damping layer} \\ \gamma_r \text{ - dimensionless damping coefficient} \end{array}$$

$$\tau(z)$$
 - damping rate (t⁻¹)
 z_d - depth of the damping layer
 γ_r - dimensionless damping coefficient

WRF Forecast over Colorado Front Range

Model Initialized 04 Dec 2007 00 UTC

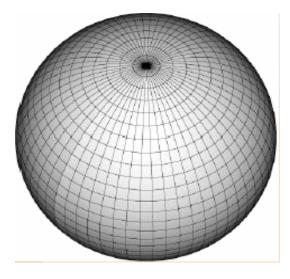


ARW Model: Coordinate Options

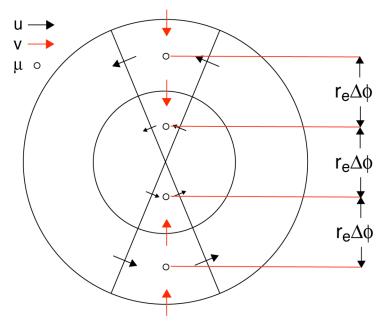
```
1. Cartesian geometry:
     idealized cases
2. Lambert Conformal:
     mid-latitude applications
3. Polar Stereographic:
     high-latitude applications
4. Mercator:
     low-latitude applications
5. Latitude-Longitude (new in ARW V3)
     global
     regional
```

Projections 1-4 are isotropic $(m_x = m_y)$ Latitude-Longitude projection is anistropic $(m_x \neq m_y)$

Global ARW - Latitude-Longitude Grid



- Map factors m_x and m_y
 - Computational grid poles need not be geographic poles.
 - Limited area and nesting capable.
- Polar boundary conditions
- Polar filtering

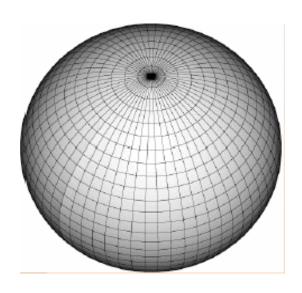


Zero meriodional flux at the poles (cell-face area is zero).

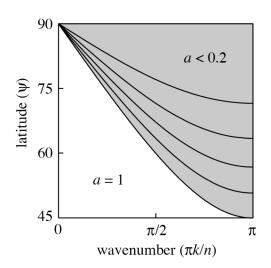
v (poles) only needed for meridional derivative of v near the poles (we interpolate).

All other meriodional derivatives are well-defined near/at poles.

ARW Filters: Polar Filter



Filter Coefficient a(k), $\psi_0 = 45^\circ$



Converging gridlines severely limit timestep. The polar filter removes this limitation.

Filter procedure - Along a grid latitude circle:

- 1. Fourier transform variable.
- 2. Filter Fourier coefficients.
- 3. Transform back to physical space.

$$\hat{\phi}(k)_{filtered} = a(k)\,\hat{\phi}(k), \quad \text{for all } k$$

$$a(k) = \min \left[1., \max \left(0., \left(\frac{\cos \psi}{\cos \psi_o} \right)^2 \frac{1}{\sin^2(\pi k/n)} \right) \right]$$

k = dimensionless wavenumber

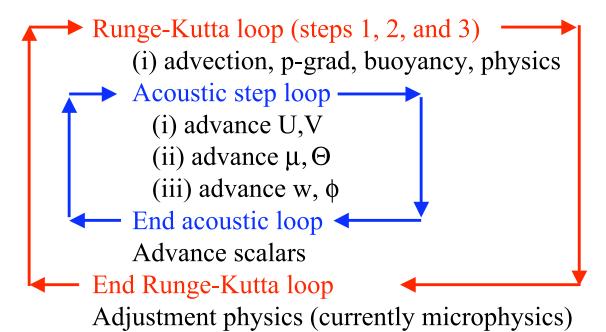
 $\hat{\phi}(k)$ = Fourier coefficients from forward transform

a(k) =filter coefficients

 $\psi = \text{ latitude } \psi_o = \text{ polar filter latitude, filter when } |\psi| > \psi_o$

WRF ARW Model Integration Procedure

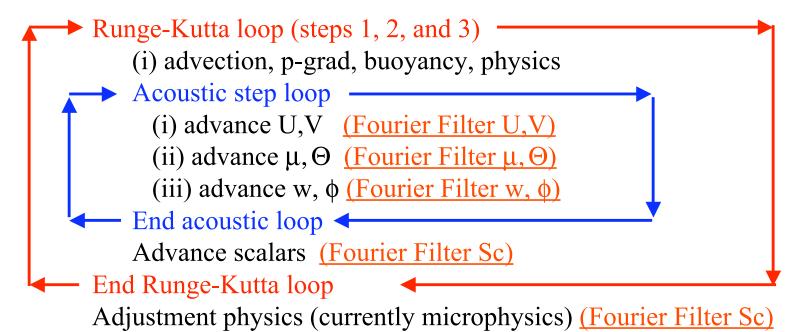
Begin time step



End time step

WRF ARW Model Integration Procedure

Begin time step



End time step

Timestep limited by minimum Δx outside of polar-filter region.

ARW Model: Boundary Condition Options

Lateral boundary conditions

- 1. Specified (Coarse grid, real-data applications).
- 2. Open lateral boundaries (gravity-wave radiative).
- 3. Symmetric lateral boundary condition (free-slip wall).
- 4. Periodic lateral boundary conditions.
- 5. Nested boundary conditions (specified).

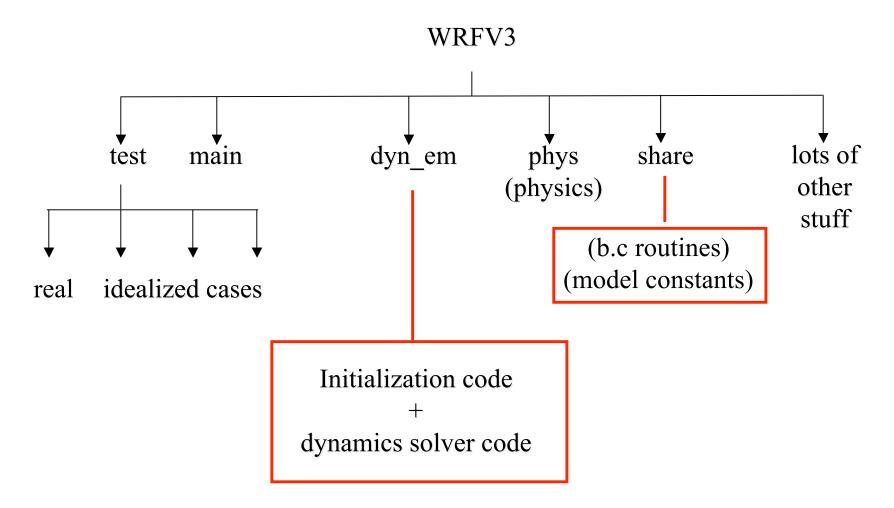
Top boundary conditions

1. Constant pressure.

Bottom boundary conditions

- 1. Free slip.
- 2. Various B.L. implementations of surface drag, fluxes.

WRF ARW code



WRF ARW Tech Note

A Description of the Advanced Research WRF Version 3 (June 2008) http://www.mmm.ucar.edu/wrf/users/pub-doc.html