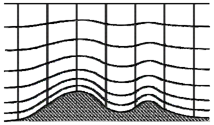
WRF Mass-Coordinate Dynamical Solver

- Terrain representation
- Vertical coordinate
- Equations / variables
- Grid staggering
- Time integration scheme
- Advection scheme
- Boundary conditions
- Nesting
- Map projections
- Dynamics parameters

Mass-Coordinate Model, Terrain Representation

Lower boundary condition for the geopotential $(\phi = gz)$ specifies the terrain elevation, and specifying the lowest coordinate surface to be the terrain results in a terrain-following coordinate.

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + \omega\frac{\partial \phi}{\partial \eta} = gw$$



Vertical coordinate:

hydrostatic pressure π $\quad \eta = \frac{(\pi - \pi_t)}{\mu}, \qquad \mu = \pi_s - \pi_t$

Flux-Form Equations in Mass Coordinate

Hydrostatic pressure coordinate:

hydrostatic pressure π

$$\eta = \frac{(\pi - \pi_t)}{\mu}, \qquad \mu = \pi_s - \pi_t$$

Conserved state variables:

$$\mu, \quad U = \mu u, \quad V = \mu v, \quad W = \mu w, \quad \Theta = \mu\theta$$

Non-conserved state variable: $\phi = gz$

Flux-Form Equations in Mass Coordinate

Inviscid, 2-D equations without rotation:

$$\frac{\partial U}{\partial t} + \mu\alpha\frac{\partial p}{\partial x} + \frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g\left(\mu - \frac{\partial p}{\partial \eta}\right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial \Omega\theta}{\partial \eta} = \mu Q$$

$$\frac{\partial \mu}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

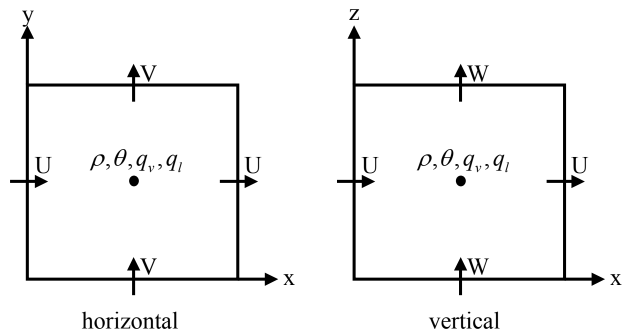
$$\frac{d\phi}{dt} = gw$$

Diagnostic relations:

$$\frac{\partial \phi}{\partial \eta} = -\mu\alpha, \qquad p = \left(\frac{R\theta}{p_0\alpha}\right)^{\gamma}, \quad \Omega = \mu\dot{\eta}$$

Height/Mass-coordinate model, grid staggering

C-grid staggering



Flux-Form Equations in Mass Coordinate

Introduce the perturbation variables:

$$\phi = \bar{\phi}(z) + \phi', \ \mu = \bar{\mu}(z) + \mu';$$
$$p = \bar{p}(z) + p', \ \alpha = \bar{\alpha}(z) + \alpha'$$

Note – $\phi = \bar{\phi}(z) = \bar{\phi}(x,y,\eta)$,

likewise $\bar{p}(x,y,\eta), \bar{\alpha}(x,y,\eta)$

Momentum and hydrostatic equations become:

$$\frac{\partial U}{\partial t} + \mu\alpha\frac{\partial p'}{\partial x} + \eta\mu\alpha'\frac{\partial \bar{\mu}}{\partial x} + \mu\frac{\partial \phi'}{\partial x} + \frac{\partial \phi'}{\partial x}\left(\frac{\partial p}{\partial \eta} - \mu'\right) = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g\left(\mu' - \frac{\partial p'}{\partial \eta}\right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \phi'}{\partial \eta} = -\bar{\mu}\alpha' - \bar{\alpha}\mu'$$

Flux-Form Equations in Mass Coordinate

Acoustic mode separation:

Recast Equations in terms of perturbation about time t

$$U' = U''^{t} + U'', \ V' = V''^{t} + V'', \ W' = W''^{t} + W'',$$
$$\Theta' = \Theta''^{t} + \Theta'', \ \mu' = \mu''^{t} + \mu'', \ \phi' = \phi''^{t} + \phi'';$$
$$p' = p''^{t} + p'', \ \alpha' = \alpha''^{t} + \alpha''$$

Linearize ideal gas law about time t

$$p'' = \frac{c_s^2}{\alpha^t}\left(\frac{\Theta''}{\Theta^t} - \frac{\alpha''}{\alpha^t} - \frac{\mu''}{\mu^t}\right)$$

$$\alpha'' = \frac{1}{\mu^t}\left(\frac{\partial \phi''}{\partial \eta} + \alpha^t \mu''\right)$$

Vertical pressure gradient becomes

$$\frac{\partial p''}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{c_s^2}{\mu^t {\alpha^t}^2}\frac{\partial \phi''}{\partial \eta} + \frac{c_s^2}{\mu^t}\frac{\Theta''}{\Theta^t}\right)$$

Flux-Form Equations in Mass Coordinate

Small (acoustic) timestep equations:

$$\delta_\tau U'' + \mu^t\alpha^t\frac{\partial p''}{\partial x} + \eta\mu^t\frac{\partial \bar{\mu}}{\partial x}\alpha'' + \mu^t\frac{\partial \phi''}{\partial x} + \frac{\partial \phi^t}{\partial x}\left(\frac{\partial p''}{\partial \eta} - \mu''\right) = R_u{}^t$$

$$\delta_\tau \mu'' + \left(\nabla \cdot \mathrm{V}''\right)_\eta^{\tau+\Delta\tau} = R_\mu{}^t$$

$$\delta_\tau \Theta'' + \left(\nabla \cdot \mathrm{V}''\theta^t\right)_\eta^{\tau+\Delta\tau} = R_\Theta{}^t$$

$$\delta_\tau W'' + g\overline{\left[\mu'' - \frac{\partial}{\partial \eta}\left(\frac{c_s^2}{\mu^t {\alpha^t}^2}\frac{\partial \phi''}{\partial \eta} + \frac{c_s^2}{\alpha^t}\frac{\Theta''}{\Theta^t}\right)\right]}^{\tau} = R_w{}^t$$

$$\delta_\tau \phi'' + \frac{1}{\mu^t}\left[\left(\mathrm{V}'' \cdot \nabla \phi^t\right)_\eta^{\tau+\Delta\tau} - g\overline{W''}^{\tau}\right] = R_\varphi{}^t$$

Acoustic Integration in the Mass Coordinate Model

Forward-backward scheme, first advance the horizontal momentum

$$\delta_\tau U'' + \mu^t \alpha^t \frac{\partial p''}{\partial x} + \eta \mu^t \frac{\partial \overline{\mu}}{\partial x}\alpha'' + \mu^t \frac{\partial \phi''}{\partial x} + \frac{\partial \phi^t}{\partial x}\left(\frac{\partial p''}{\partial \eta} - \mu''\right) = R_u^{\,t}$$

Second, advance continuity equation, diagnose omega, and advance thermodynamic equation

$$\delta_\tau \mu'' + \left(\nabla \cdot \mathrm{V}''\right)_\eta^{\tau+\Delta\tau} = R_\mu^{\,t}$$

$$\delta_\tau \Theta'' + \left(\nabla \cdot \mathrm{V}''\theta^t\right)_\eta^{\tau+\Delta\tau} = R_\Theta^{\,t}$$

Finally, vertically-implicit integration of the acoustic and gravity wave terms

$$\delta_\tau W'' + g\,\overline{\left[\mu'' - \frac{\partial}{\partial \eta}\left(\frac{c_s^2}{\mu^t {\alpha^t}^2}\frac{\partial \phi''}{\partial \eta} + \frac{c_s^2}{\alpha^t}\frac{\Theta''}{\Theta^t}\right)\right]}^{\tau} = R_w^{\,t}$$

$$\delta_\tau \phi'' + \frac{1}{\mu^t}\left[\left(\mathrm{V}''\cdot\nabla\phi^t\right)_\eta^{\tau+\Delta\tau} - g\overline{W''}^{\tau}\right] = R_\phi^{\,t}$$

Moist Equations in Mass-Coordinate Model

Moist Equations:

$$\frac{\partial U}{\partial t} + \alpha\mu_d\frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{\partial(\mu_d q_{v,l})}{\partial t} + \frac{\partial U(\mu_d q_{v,l})}{\partial x} + \frac{\partial \Omega(\mu_d q_{v,l})}{\partial \eta} = \mu Q_{v,l}$$

Diagnostic relations:

$$\frac{\partial \phi}{\partial \eta} = -\alpha_d\mu_d, \qquad p = \left(\frac{R\Theta}{p_o\mu_d\alpha_v}\right)^{\gamma}$$

Height/Mass-Coordinate Model, Time Integration

3rd Order Runge-Kutta time integration

advance $\phi^t \rightarrow \phi^{t+1}$

$$\phi^* = \phi^t + \frac{\Delta t}{3}R\left(\phi^t\right)$$

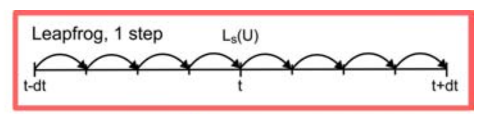
$$\phi^{**} = \phi^t + \frac{\Delta t}{2}R\left(\phi^*\right)$$

$$\phi^{t+1} = \phi^t + \frac{\Delta t}{3}R\left(\phi^{**}\right)$$

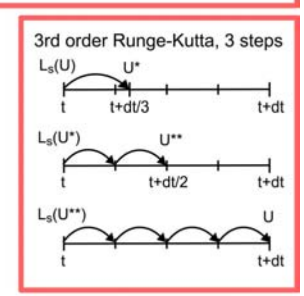
Amplification factor $\phi_t = ik\phi;\quad \phi^{n+1} = A\phi^n;\quad |A| = 1 - \frac{(k\Delta t)^4}{24}$

Time-Split Leapfrog and Runge-Kutta Integration Schemes

Integrate
$U_t = L_f(U) + L_s(U)$



- LF is formally 1st order accurate, RK3 is 3rd order accurate
- RK3 stable for centered and upwind advection schemes, LF only stable for centered schemes.
- RK3 is stable for timesteps 2 to 3 times larger than LF.
- LF requires only one advection evaluation per timestep, RK3 requires three per timestep.



WRF Mass-Coordinate Model Integration Procedure

Begin time step

- Runge-Kutta loop (steps 1, 2, and 3)
  - (i) advection, p-grad, buoyancy using $\left(\phi^t, \phi^*, \phi^{**}\right)$
  - (ii) physics if step 1, save for steps 2 and 3
  - (iii) mixing, other non-RK dynamics, save…
  - (iv) assemble dynamics tendencies
  - Acoustic step loop
    - (i) advance U,V, then μ, Θ, then w, ϕ
    - (ii) time-average U,V, Ω
  - End acoustic loop
  - Advance scalars using time-averaged U,V, Ω
- End Runge-Kutta loop
- Adjustment physics (currently microphysics)

End time step

Advection in the Height/Mass Coordinate Model

2nd, 3rd, 4th, 5th and 6th order centered and upwind-biased schemes are available in the WRF model.

Example: 5th order scheme

$$\frac{\partial(U\phi)}{\partial x} = \frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(U\phi) - F_{i-\frac{1}{2}}(U\phi)\right)$$

where

$$F_{i-\frac{1}{2}}(U\phi) = U_{i-\frac{1}{2}}\left\{\frac{37}{60}\left(\phi_i + \phi_{i-1}\right) - \frac{2}{15}\left(\phi_{i+1} + \phi_{i-2}\right) + \frac{1}{60}\left(\phi_{i+2} + \phi_{i-3}\right)\right\}$$

$$- sign(1,U)\frac{1}{60}\left\{\left(\phi_{i+2} - \phi_{i-3}\right) - 5\left(\phi_{i+1} - \phi_{i-2}\right) + 10\left(\phi_i - \phi_{i-1}\right)\right\}$$

Advection in the Height/Mass Coordinate Model

For constant U, the 5th order flux divergence tendency becomes

$$\Delta t \left.\frac{\delta(U\phi)}{\Delta x}\right|_{5th} = \Delta t \left.\frac{\delta(U\phi)}{\Delta x}\right|_{6th} - \underbrace{\left|\frac{U\Delta t}{\Delta x}\right| \frac{1}{60}\left(-\phi_{i-3} + 6\phi_{i-2} - 15\phi_{i-1} + 20\phi_i - 15\phi_{i+1} + 6\phi_{i+2} - \phi_{i+3}\right)}_{\frac{Cr}{60}\frac{\partial^6 \phi}{\partial x^6} + H.O.T}$$

The odd-ordered flux divergence schemes are equivalent to the next higher ordered (even) flux-divergence scheme plus a dissipation term of the higher even order with a coefficient proportional to the Courant number.

Mass Coordinate Model: Boundary Condition Options

Lateral boundary conditions

1. Specified (Coarse grid, real-data applications).
2. Open lateral boundaries (gravity-wave radiative).
3. Symmetric lateral boundary condition (free-slip wall).
4. Periodic lateral boundary conditions.
5. Nested boundary conditions (specified).

Top boundary conditions

1. Constant pressure.
2. Gravity-wave radiative condition (not yet implemented).
3. Absorbing upper layer (increased horizontal diffusion).
4. Rayleigh damping upper layer (not yet implemented).

Bottom boundary conditions

1. Free slip.
2. Various B.L. implementations of surface drag, fluxes.

Mass Coordinate Model: Nesting

2-way nesting

1. Multiple domains run concurrently
2. Multiple levels, multiple nests per level
3. Any integer ratio grid size and time step
4. Parent domain provides nest boundaries
5. Nest feeds back interior values to parent

1-way nesting

1. Parent domain is run first
2. *ndown* uses coarse output to generate nest boundary conditions
3. Nest initial conditions from fine-grid input file
4. Nest is run after *ndown*

Note: In V2.0, nest has to start at same time as coarse mesh if using fine-scale input file, but can start any time if just using interpolation from coarse mesh to initialize it in 2-way nesting.

Mass/Height Coordinate Model: Coordinate Options

1. Cartesian geometry (idealized cases)
2. Lambert Conformal
3. Polar Stereographic
4. Mercator

Mass Coordinate Model: Dynamics Parameters

3rd order Runge-Kutta time step

Courant number limited, 1D: $C_r = \frac{U \Delta t}{\Delta x} < 1.73$

Generally stable using a timestep approximately twice as large as used in a leapfrog model.

Acoustic time step

2D horizontal Courant number limited: $C_r = \frac{C_s \Delta \tau}{\Delta h} < \frac{1}{\sqrt{2}}$

$\Delta \tau_{sound} = \Delta t_{RK} / (\text{number of acoustic steps})$

Guidelines for time step

dt in seconds should be about 6**dx/1000* (grid size in kilometers). Larger *dt* can be used in smaller-scale dry situations, but *time_step_sound* (=4) should increase proportionately if >6 factor is used.

Mass Coordinate Model: Dynamics Parameters

Divergence damping coefficient: smdiv=0.1 recommended.

External mode damping coefficient: emdiv=0.01 recommended.

Vertically-implicit off-centering parameter: epssm=0.1 recommended.

Advection scheme order: 5^{th} order horizontal, 3^{rd} order vertical recommended.