The Advanced Research WRF (ARW) Dynamics Solver

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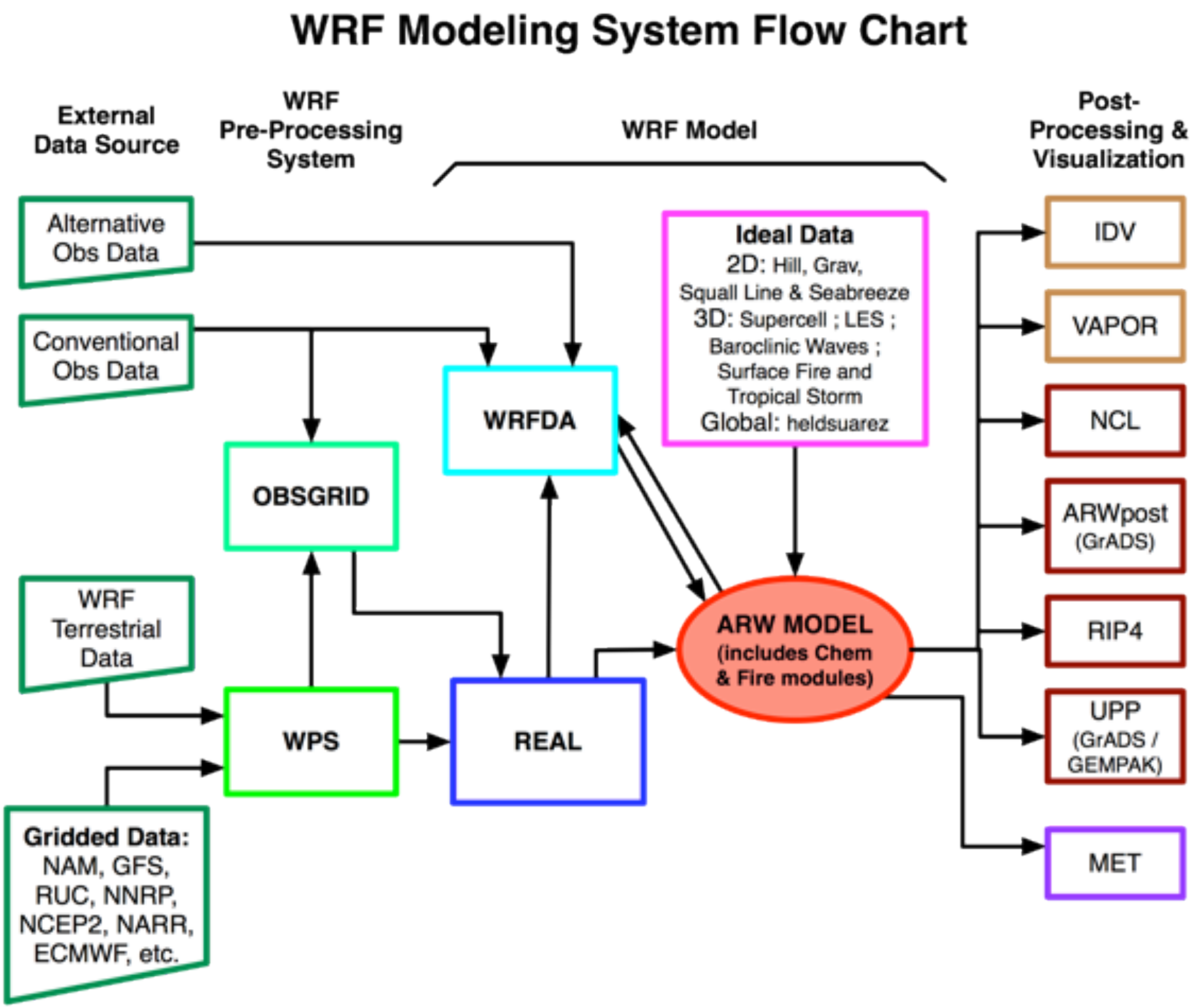


WRF ARW Tech Note

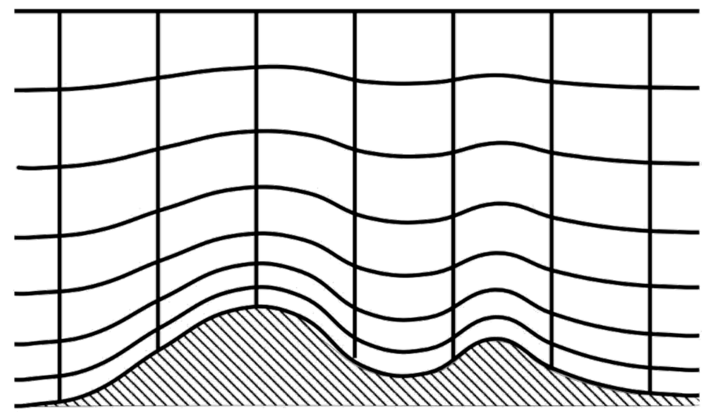
A Description of the Advanced Research WRF Version 3

http://www.mmm.ucar.edu/wrf/users/pub-doc.html

ARW Dynamical Solver

- Terrain representation
- Vertical coordinate
- Equations / variables
- Time integration scheme
- Grid staggering
- Advection scheme
- Time step parameters
- Filters
- Boundary conditions
- Nesting
- Map projections

Vertical Coordinate and Prognostic Variables



Hydrostatic pressure π

Column mass $\mu = \pi_s - \pi_t$
(per unit area)

Vertical coordinate $\eta = \dfrac{(\pi - \pi_t)}{\mu}$

Layer mass $\mu \Delta \eta = \Delta \pi = g \rho \Delta z$
(per unit area)

Conserved state (prognostic) variables:

$$\mu, \quad U = \mu u, \quad V = \mu v, \quad W = \mu w, \quad \Theta = \mu \theta$$

Non-conserved state variable: $\phi = gz$

2D Flux-Form Moist Equations in ARW

Moist Equations:

$$\frac{\partial U}{\partial t} + \alpha\mu_d \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial \Omega\theta}{\partial \eta} = \mu Q$$

$$\frac{d\phi}{dt} = gw$$

$$\frac{\partial(\mu_d q_{v,l})}{\partial t} + \frac{\partial(U q_{v,l})}{\partial x} + \frac{\partial(\Omega q_{v,l})}{\partial \eta} = \mu Q_{v,l}$$

Diagnostic relations:

$$\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, \qquad p = \left(\frac{R\Theta}{p_o \mu_d \alpha_v}\right)^{\gamma}$$

Time Integration in ARW

3rd Order Runge-Kutta time integration

advance $\phi^{t} \rightarrow \phi^{t+\Delta t}$

$$\phi^{*} = \phi^{t} + \frac{\Delta t}{3} R\left(\phi^{t}\right)$$

$$\phi^{**} = \phi^{t} + \frac{\Delta t}{2} R\left(\phi^{*}\right)$$

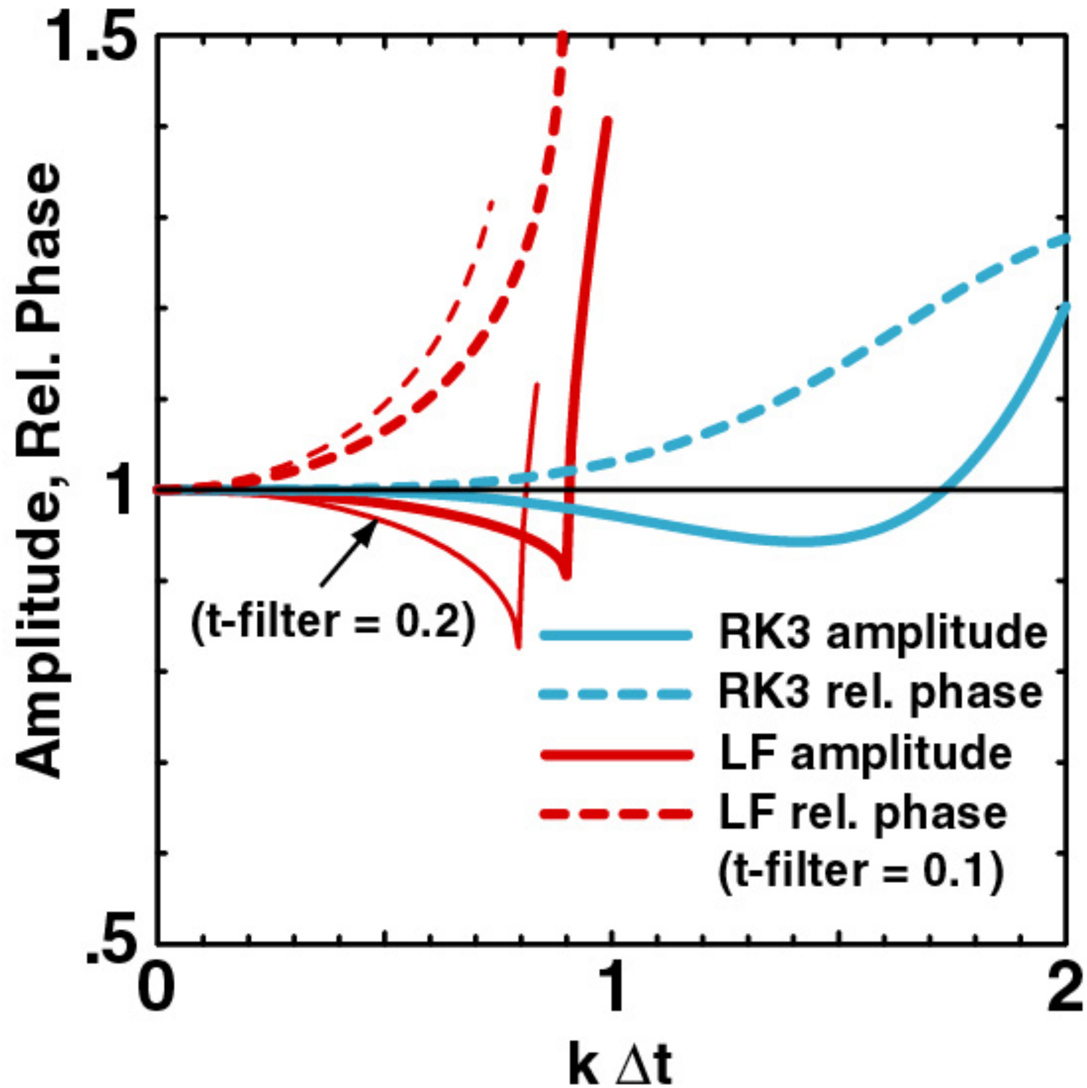
$$\phi^{t+\Delta t} = \phi^{t} + \Delta t\, R\left(\phi^{**}\right)$$

Amplification factor $\phi_{t} = i k \phi; \quad \phi^{n+1} = A \phi^{n}; \quad |A| = 1 - \frac{(k \Delta t)^{4}}{24}$

Phase and amplitude errors for LF, RK3

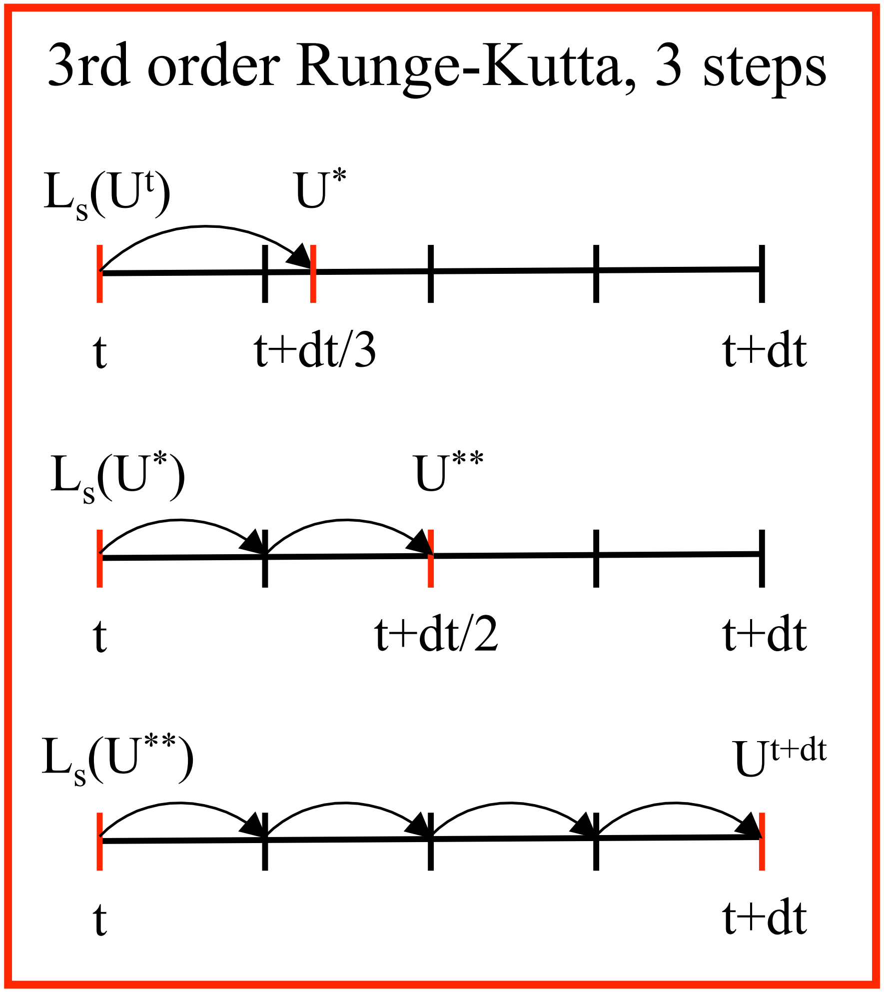
Oscillation equation analysis

$\phi_t = ik\phi$



Time-Split Runge-Kutta Integration Scheme

$$U_t = L_{fast}(U) + L_{slow}(U)$$



- RK3 is 3rd order accurate for linear eqns, 2nd order accurate for nonlinear eqns.
- Stable for centered and upwind advection schemes.
- Stable for Courant number $Udt/dx < 1.73$
- Three $L_{slow}(U)$ evaluations per timestep.

WRF ARW Model Integration Procedure

Begin time step

Runge-Kutta loop (steps 1, 2, and 3)

(i) advection, p-grad, buoyancy using ([illegible]^t, [illegible]*, [illegible]**)

(ii) physics if step 1, save for steps 2 and 3

(iii) mixing, other non-RK dynamics, save…

(iv) assemble dynamics tendencies

Acoustic step loop

(i) advance U,V, then m, Q, then w, *f*

(ii) time-average U,V, W

End acoustic loop

Advance scalars using time-averaged U,V, W

End Runge-Kutta loop

Adjustment physics (currently microphysics)

End time step

Flux-Form Perturbation Equations

Introduce the perturbation variables:

$$\phi = \overline{\phi}(\overline{z}) + \phi', \mu = \overline{\mu}(\overline{z}) + \mu';$$
$$p = \overline{p}(\overline{z}) + p', \alpha = \overline{\alpha}(\overline{z}) + \alpha'$$

Note – $\phi = \overline{\phi}(\overline{z}) = \overline{\phi}(x, y, \eta),$

likewise $\overline{p}(x, y, \eta), \overline{\alpha}(x, y, \eta)$

Reduces horizontal pressure-gradient errors.

For small time steps, recast variables as perturbations from time t

$$U' = U'^{t} + U'', \ V' = V'^{t} + V'', \ W' = W'^{t} + W'',$$
$$\Theta' = \Theta'^{t} + \Theta'', \ \mu' = \mu'^{t} + \mu'', \ \phi' = \phi'^{t} + \phi'';$$
$$p' = p'^{t} + p'', \ \alpha' = \alpha'^{t} + \alpha''$$

Allows vertical pressure gradient to be expressed in terms of ”.

Small Time Step Integration of Acoustic/Gravity Wave Terms

(Without expanding variables into perturbation form)

$U^{\tau+\Delta\tau}$

$$\frac{\partial U}{\partial t} + \left(\mu_d \alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \phi}{\partial x} \right)^{\tau} = R_U^t$$

$\mu_d^{\tau+\Delta\tau}$ Ω^{τ}

$$\frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x}^{\tau+\Delta\tau} + \frac{\partial \Omega}{\partial \eta}^{\tau+\Delta\tau} = 0$$

$\Theta^{\tau+\Delta\tau}$

$$\frac{\partial \Theta}{\partial t} + \left(\frac{\partial U \theta^t}{\partial x} + \frac{\partial \Omega \theta^t}{\partial \eta} \right)^{\tau+\Delta\tau} = R_\Theta^t$$

$W^{\tau+\Delta\tau}$

$$\frac{\partial W}{\partial t} + g \overline{\left(\mu_d - \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \right)}^{\tau} = R_W^t$$

$\phi^{\tau+\Delta\tau}$

$$\mu_d^t \frac{\partial \phi}{\partial t} + U^{\tau+\Delta\tau} \frac{\partial \phi}{\partial x}^t + \Omega^{\tau+\Delta\tau} \frac{\partial \phi}{\partial \eta}^t - g \overline{W}^{\tau} = R_\phi^t$$

- Forward-backward differencing on U, □, and □ equations
- Vertically implicit differencing on W and □ equations

Hydrostatic Option

Instead of solving vertically implicit equations for *W* and [illegible]

Integrate the hydrostatic equation to obtain *p* ([illegible]):

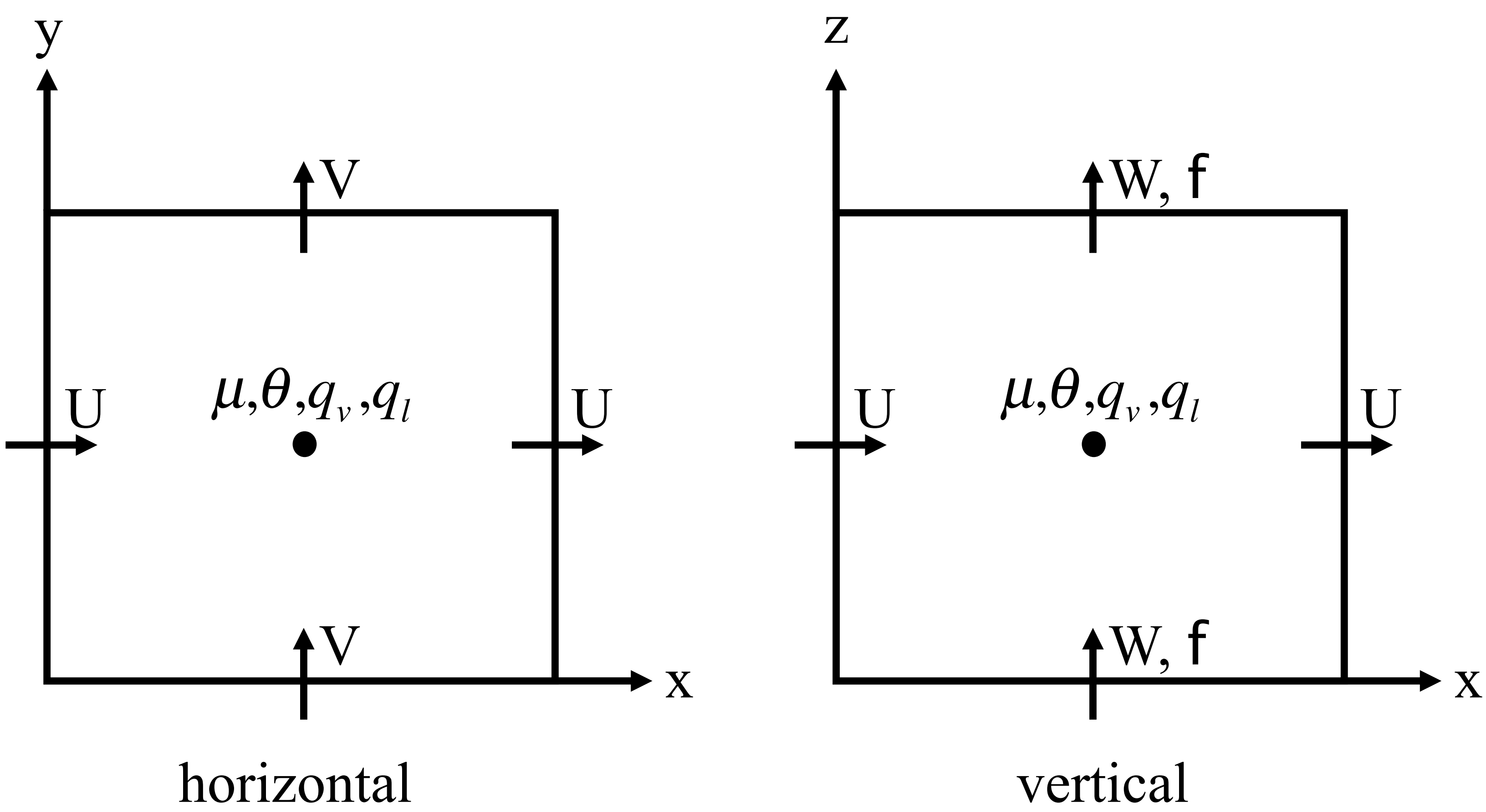
$$\frac{\partial p}{\partial \eta} = \left(\frac{\alpha_d}{\alpha}\right)^t \mu$$

Recover [illegible] and [illegible] fron $p = p_0\left(\frac{R\theta}{p_0\alpha_v}\right)^\gamma$, an $\frac{\partial \phi}{\partial \eta} = -\mu_d\alpha_d$

W is no longer required during the integration.

ARW model, grid staggering

C-grid staggering



Advection in the ARW Model

2nd, 3rd, 4th, 5th and 6th order centered and upwind-biased schemes are available in the ARW model.

Example: 5th order scheme

$$\frac{\partial (U\psi)}{\partial x} = \frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(U\psi) - F_{i-\frac{1}{2}}(U\psi) \right)$$

where

$$F_{i-\frac{1}{2}}(U\psi) = U_{i-\frac{1}{2}}\left\{ \frac{37}{60}(\psi_i + \psi_{i-1}) - \frac{2}{15}(\psi_{i+1} + \psi_{i-2}) + \frac{1}{60}(\psi_{i+2} + \psi_{i-3}) \right\}$$

$$-sign(1,U)\,\frac{1}{60}\left\{ (\psi_{i+2} - \psi_{i-3}) - 5(\psi_{i+1} - \psi_{i-2}) + 10(\psi_i - \psi_{i-1}) \right\}$$

Advection in the ARW Model

For constant U, the 5[th] order flux divergence tendency becomes

$$\Delta t \frac{\delta(U\psi)}{\Delta x}\bigg|_{5th} = \Delta t \frac{\delta(U\psi)}{\Delta x}\bigg|_{6th} - \underbrace{\left|\frac{U\Delta t}{\Delta x}\right| \frac{1}{60}\left(-\psi_{i-3} + 6\psi_{i-2} - 15\psi_{i-1} + 20\psi_i - 15\psi_{i+1} + 6\psi_{i+2} - \psi_{i+3}\right)}_{\frac{Cr}{60}\frac{\partial^6 \psi}{\partial x^6} + H.O.T}$$

The odd-ordered flux divergence schemes are equivalent to the next higher ordered (even) flux-divergence scheme plus a dissipation term of the higher even order with a coefficient proportional to the Courant number.

Maximum Courant Number for Advection

$$C_a = U \Delta t / \Delta x$$

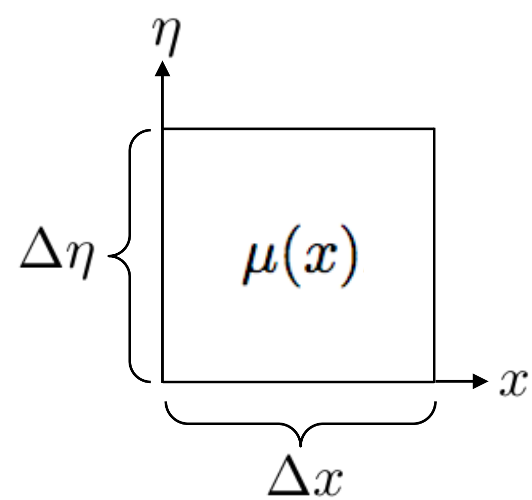
Time Integration Scheme	*Advection Scheme*				
	2nd	*3rd*	*4th*	*5th*	*6th*
Leapfrog (g=0.1)	0.91	U	0.66	U	0.57
RK2	U	0.90	U	0.39	U
RK3	1.73	1.63	1.26	1.43	1.09

U = unstable

(Wicker & Skamarock, 2002)

Mass Conservation in the ARW Model

control volume
(2D example)



Mass in a control volume is proportional to

$$(\Delta x \Delta \eta)(\mu)^t$$

since $\mu(x)\Delta\eta = \Delta\pi = -g\rho\Delta z$

Mass Conservation in the ARW Model

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

2D example

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

Change in mass over a time step

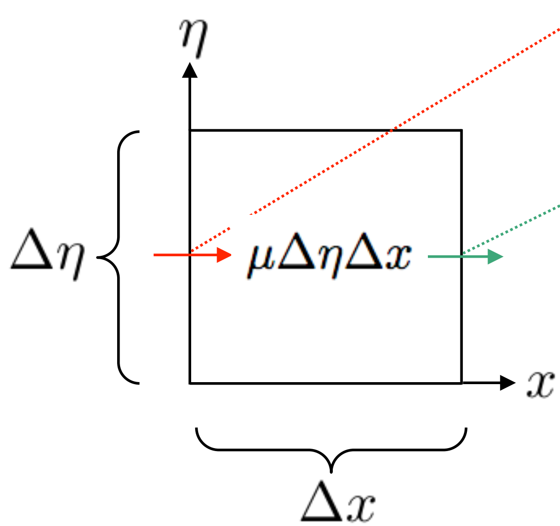
mass fluxes through control volume faces

Mass Conservation in the ARW Model

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta)\cdot\left[(\mu)^{t+\Delta t}-(\mu)^{t}\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta}\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2}\right]$$



Horizontal fluxes through the vertical control-volume faces

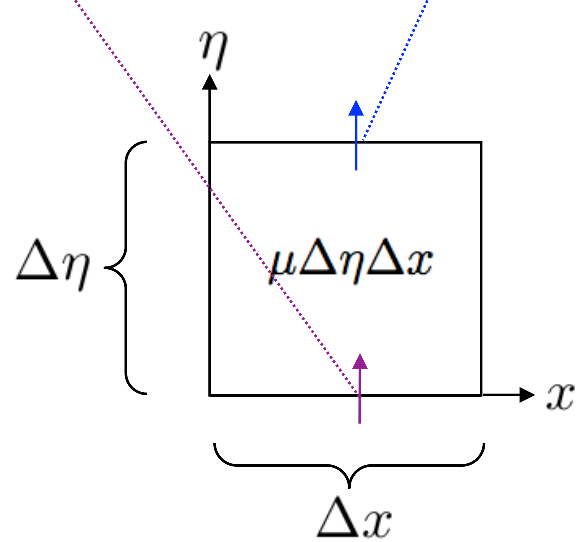
Mass Conservation in the ARW Model

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Mass conservation equation

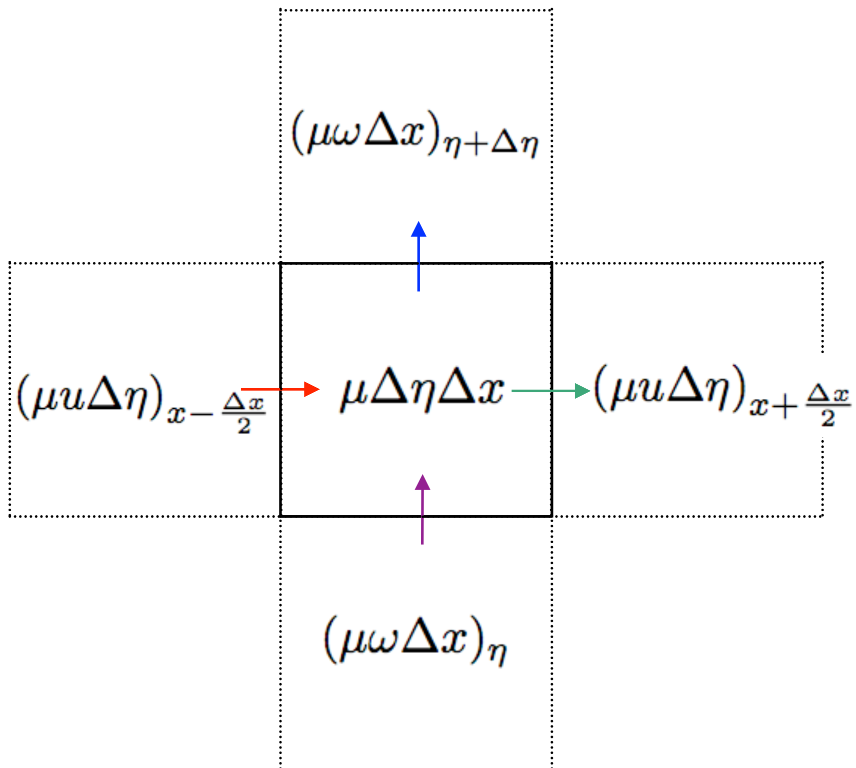
$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

Vertical fluxes through the horizontal control-volume faces



Mass Conservation in the ARW Model

The same mass fluxes are used for neighboring grid cells - hence mass is conserved locally and globally.



Scalar Mass Conservation in the ARW Model

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Scalar mass $(\Delta x \Delta \eta)(\mu\phi)^t$

Mass conservation equation:

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

change in mass over a time step (left side); mass fluxes through control volume faces (right side)

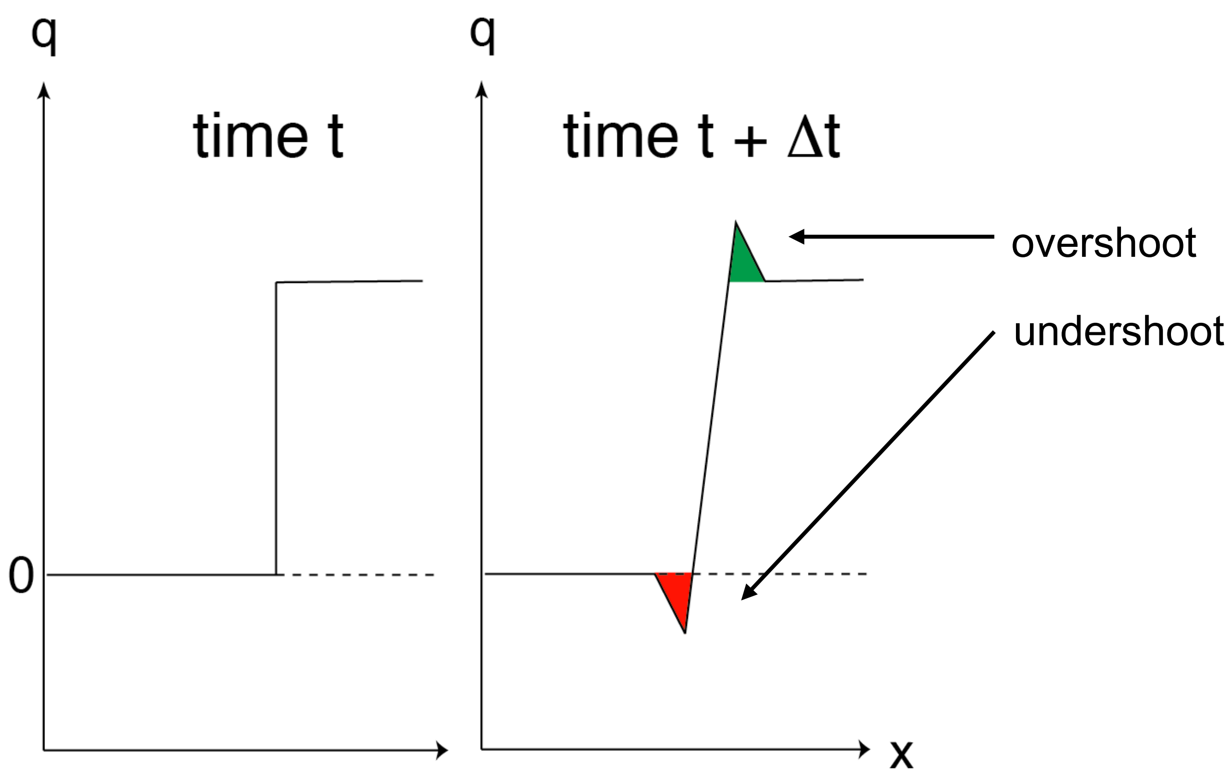
Scalar mass conservation equation:

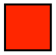
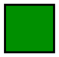
$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu\phi)^{t+\Delta t} - (\mu\phi)^t)\right] = \left[(\mu u \phi \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \phi \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \phi \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \phi \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

change in tracer mass over a time step (left side); tracer mass fluxes through control volume faces (right side)

Moisture Transport in ARW

1D advection



ARW scheme is conservative,
but not positive definite nor monotonic.
Removal of negative q 
results in spurious source of q  .

Positive-Definite/Monotonic Flux Renormalization

Scalar update, last RK3 step

$$(\mu\phi)^{t+\Delta t} = (\mu\phi)^t - \Delta t \sum_{i=1}^{n} \delta_{x_i}[f_i] \qquad (1)$$

(1) Decompose flux: $f_i = f_i^{upwind} + f_i^c$

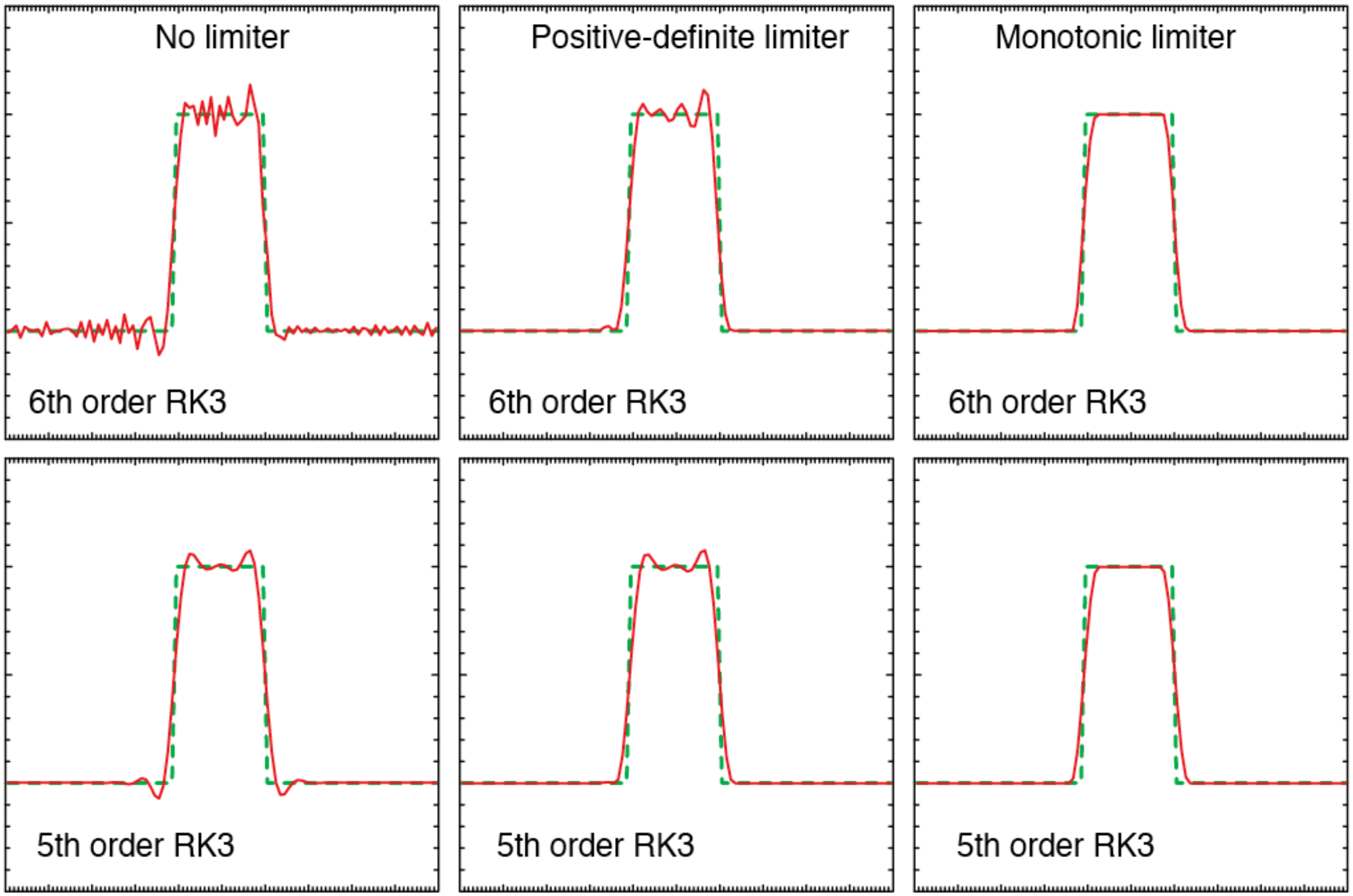
(2) Renormalize high-order correction fluxes f_i^c such that solution is positive definite or monotonic: $f_i^c = R(f_i^c)$

(3) Update scalar eqn. (1) using $f_i = f_i^{upwind} + R(f_i^c)$

Skamarock, MWR 2006, 2241-2250

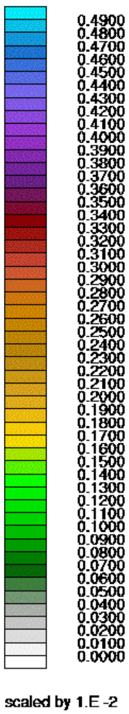
PD/Monotonic Limiters in ARW - 1D Example
Top-Hat Advection

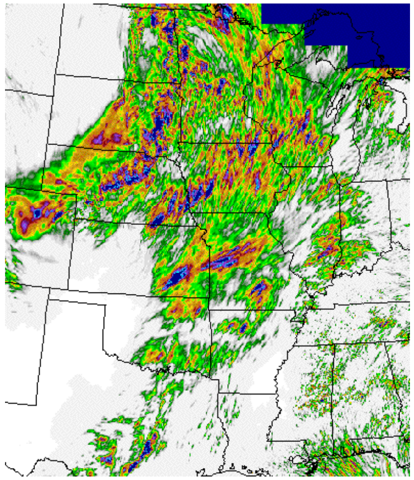
1D Top-hat transport   Cr = 0.5, 1 revolution, 200 steps



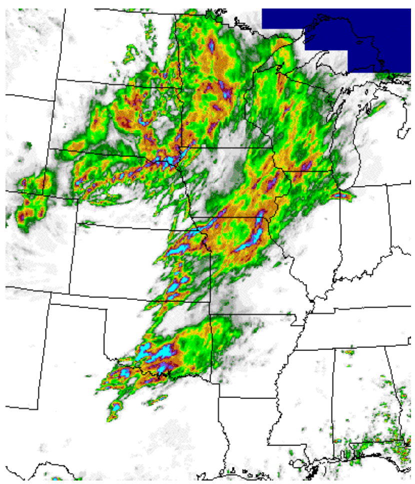
Moisture Transport in ARW: High Precipitation Bias

Initialized 00 UTC 04 June 2005



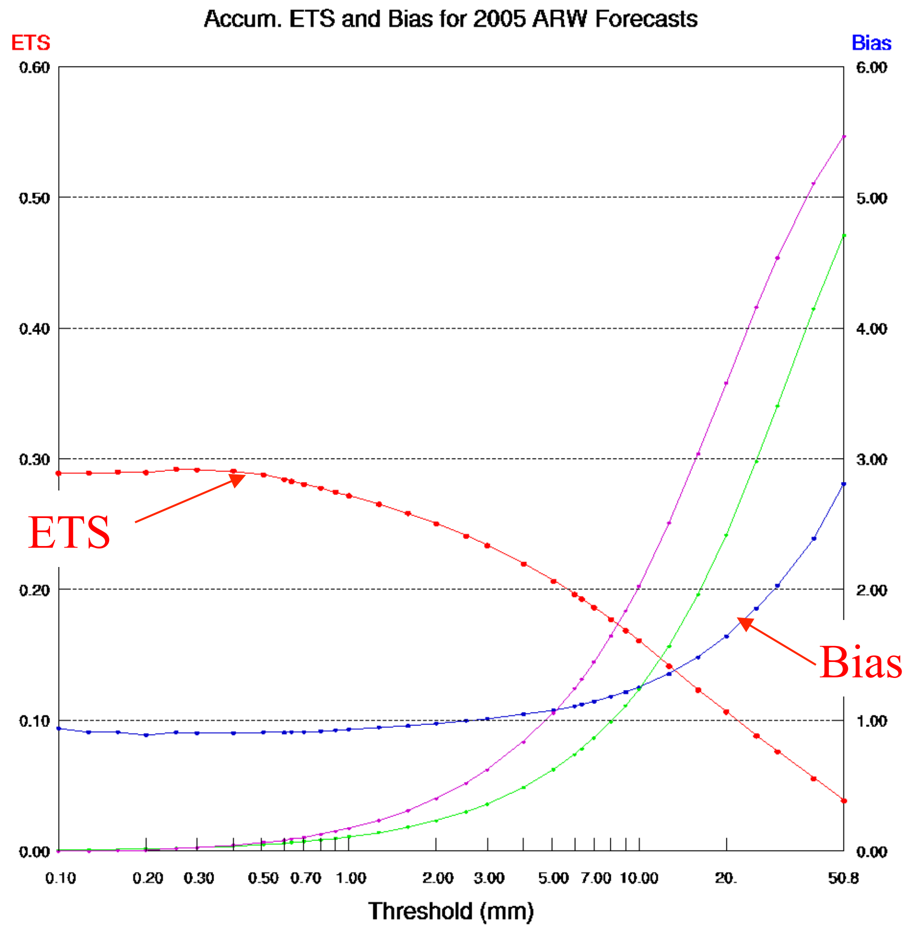


ARW 12-36 h Forecast Precip



24 h Observed (ST4) Precip

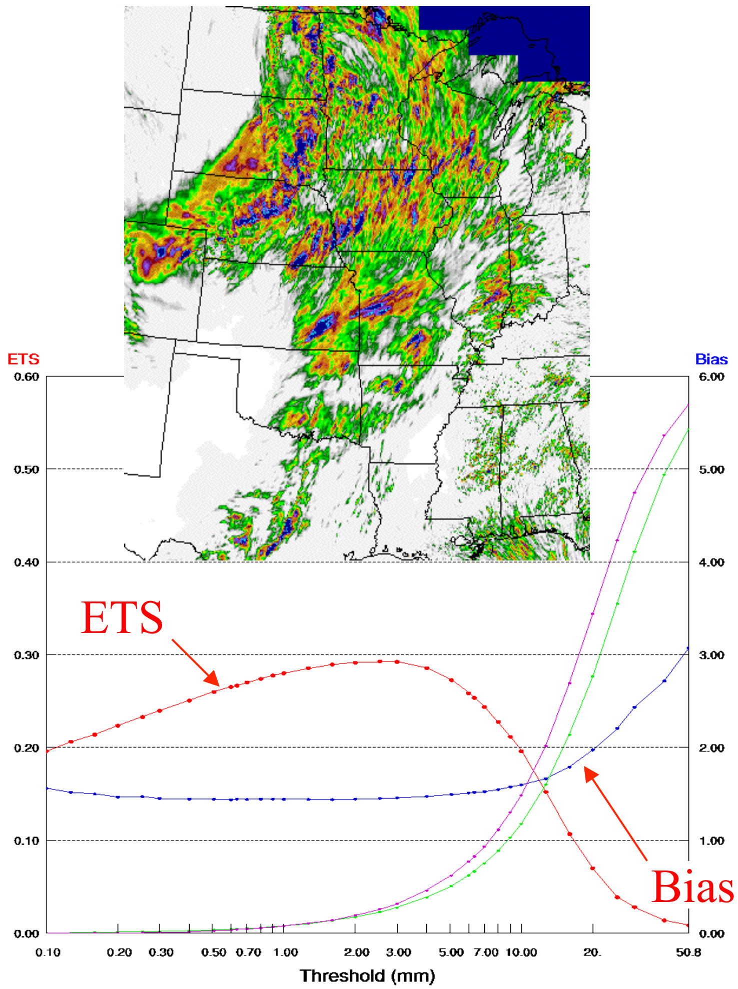
2005 ARW 4 km Forecasts:



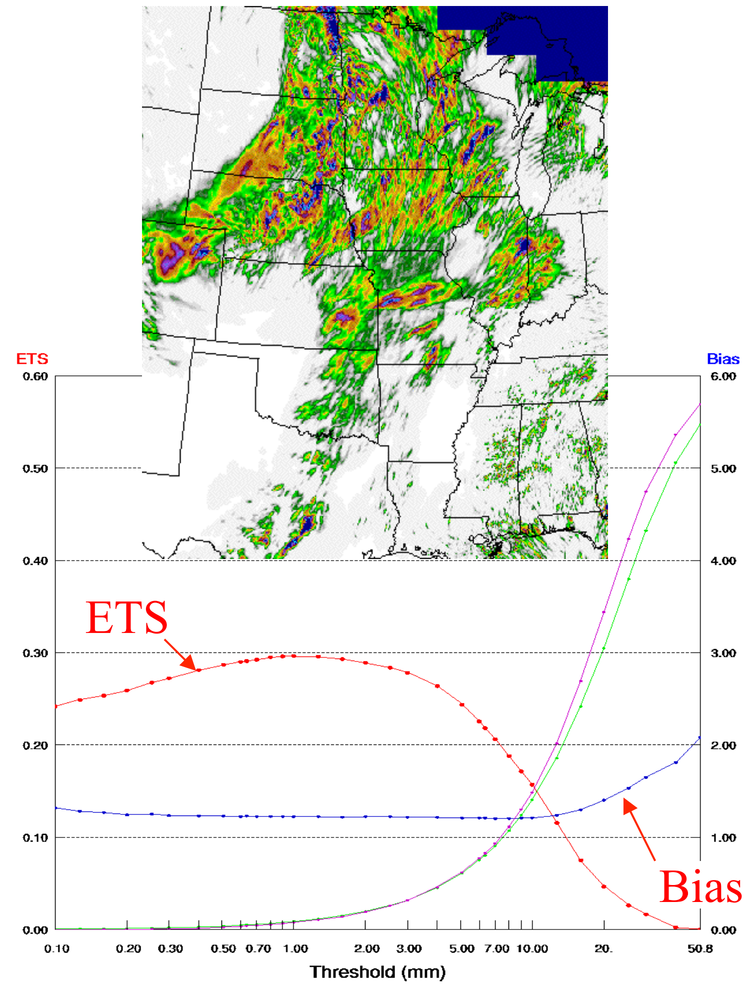
Moisture Transport in ARW: 24 h ETS and BIAS

Initialized 00 UTC 04 June 2005

Standard advection



Positive-definite advection



ARW Model: Dynamics Parameters

3rd order Runge-Kutta time step

Courant number limited, 1D: $C_r = \frac{U\Delta t}{\Delta x} < 1.43$

Generally stable using a timestep approximately twice as large as used in a leapfrog model.

Acoustic time step

2D horizontal Courant number limited: $C_r = \frac{C_s \Delta\tau}{\Delta h} < \frac{1}{\sqrt{2}}$

$$\Delta\tau_{sound} = \Delta t_{RK} / (\text{number of acoustic steps})$$

Guidelines for time step

Δt in seconds should be about 6*Δx (grid size in kilometers). Larger Δt can be used in smaller-scale dry situations, but *time_step_sound* (default = 4) should increase proportionately if larger Δt is used.

ARW Filters: Divergence Damping

Purpose: filter acoustic modes (3-D divergence, $D = \nabla \cdot \rho \mathbf{V}$ *)*

$$\left\{ \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla p + \ldots = \gamma_d' \nabla D \right\}$$

$$\nabla \cdot \left\{ \quad \right\} \quad \rightarrow \quad \frac{\partial D}{\partial t} + \nabla^2 p + \ldots = \gamma_d' \nabla^2 D$$

From the pressure equation: $p_t \simeq c^2 D$

$$\boxed{\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla [p_\tau + \gamma_d (p^\tau - p^{\tau - \Delta \tau})] + \ldots = 0}$$

γ_d = 0.1 recommended (default)

(Illustrated in height coordinates for simplicity)

ARW Filters: Vertically Implicit Off-Centered Acoustic Step

Purpose: damp vertically-propagating acoustic modes

$$\frac{\partial W}{\partial t} + g\overline{\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right)}^{\tau} = \ldots$$

$$\frac{\partial \phi}{\partial t} - \frac{g}{\mu_d^t}\overline{W}^{\tau} = \ldots$$

$$\overline{(\ \ )}^{\tau} = \frac{1+\beta}{2}\overline{(\ \ )}^{\tau+\Delta\tau} + \frac{1-\beta}{2}\overline{(\ \ )}^{\tau}$$

Slightly forward centering the vertical pressure gradient damps 3-D divergence as demonstrated for the divergence damper

β = 0.1 recommended (default)

ARW Filters: External Mode Filter

Purpose: filter the external mode

Vertically integrated horizontal divergence, $D_h = \int_1^0 (\nabla_\eta \cdot \mu \mathbf{V}_h) d\eta$

$$\left\{ \frac{\partial \mu \mathbf{V}_h}{\partial t} + \ldots = -\gamma_e \nabla_\eta D_h \right\}$$

$$\int_1^0 \nabla_\eta \cdot \left\{ \quad \right\} d\eta \quad \rightarrow \quad \frac{\partial D_h}{\partial t} + \ldots = \gamma_e \nabla^2 D_h$$

Continuity equation: $\frac{\partial \mu}{\partial t} = -\nabla_\eta \cdot \mu \mathbf{V}_h - \frac{\partial \mu \dot{\eta}}{\partial \eta} = D_h$

$$\frac{\partial \mu \mathbf{V}_h}{\partial \tau} + \ldots = -\gamma_e \frac{\Delta x^2}{\Delta \tau^2} \nabla_\eta (\mu^\tau - \mu^{\tau - \Delta\tau})$$

γ_e = 0.01 recommended (default)

(Primarily for real-data applications)

ARW Filters: Vertical Velocity Damping

Purpose: damp anomalously-large vertical velocities

(usually associated with anomalous physics tendencies)

Additional term:

$$\partial_t W = \ldots - \mu_d \, \mathrm{sign}(W) \gamma_w (Cr - Cr_\beta)$$

$$Cr = \left| \frac{\Omega dt}{\mu d\eta} \right|$$

Cr_β [illegible] typical value (default)

γ_w = 0.3 m/s^2 recommended (default)

ARW Filters: 2nd-Order Horizontal Mixing, Horizontal-Deformation-Based K_h

Purpose: mixing on horizontal coordinate surfaces
(real-data applications)

$$K_h = C_s^2\, l^2 \left[0.25 (D_{11} - D_{22})^2 + \overline{D_{12}^2}^{xy} \right]^{\frac{1}{2}}$$

where $$l = (\Delta x \Delta y)^{1/2}$$

$$D_{11} = 2\, m^2 \left[\partial_x (m^{-1} u) - z_x \partial_z (m^{-1} u) \right]$$
$$D_{22} = 2\, m^2 \left[\partial_y (m^{-1} v) - z_y \partial_z (m^{-1} v) \right]$$
$$D_{12} = m^2 \left[\partial_y (m^{-1} u) - z_y \partial_z (m^{-1} u) + \partial_x (m^{-1} v) - z_x \partial_z (m^{-1} v) \right]$$

$C_s = 0.25$ (Smagorinsky coefficient, default value)

Implicit Rayleigh w Damping Layer for Split-Explicit Nonhydrostatic NWP Models (gravity-wave absorbing layer)

Modification to small time step:

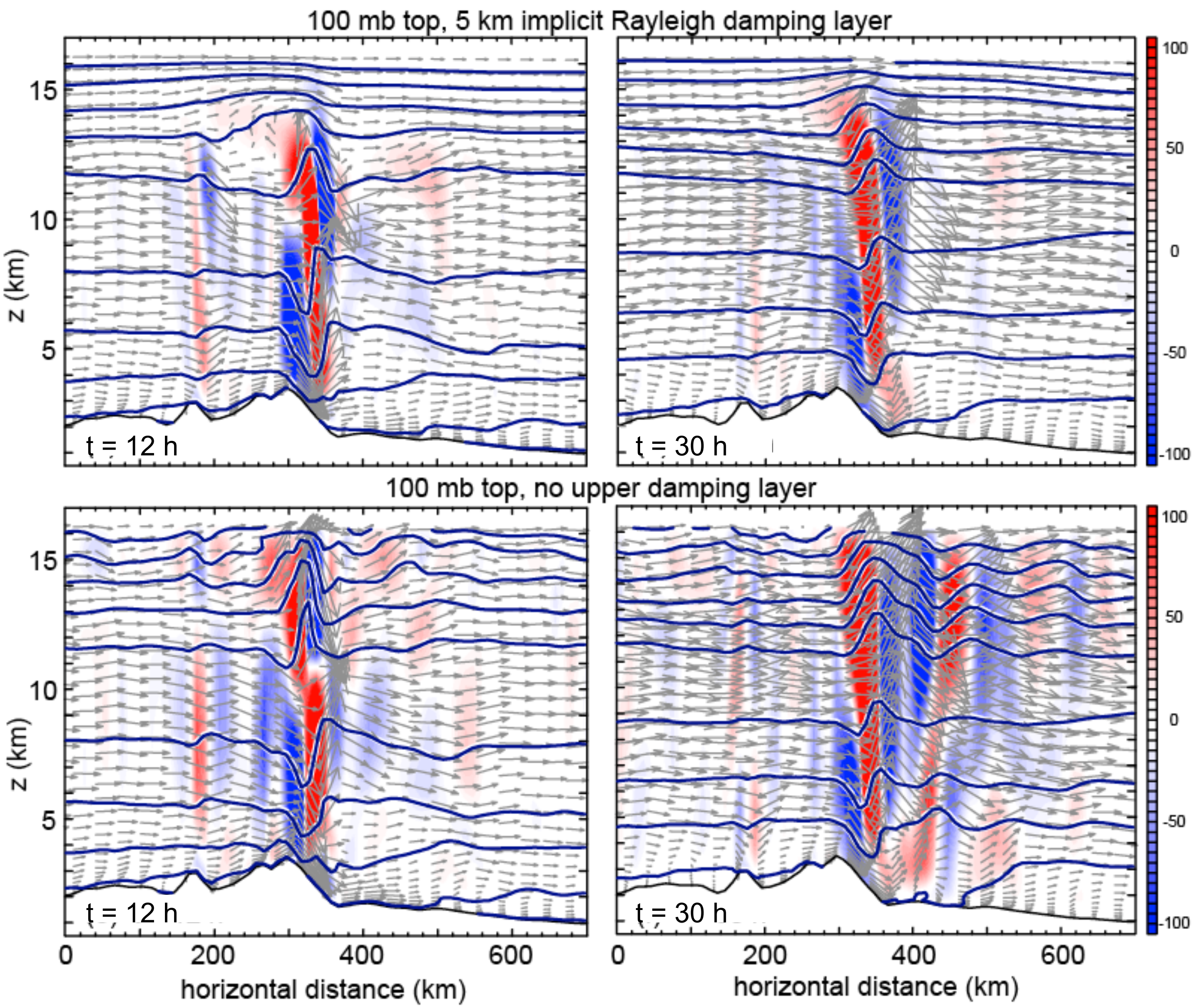
- Step horizontal momentum, continuity, and potential temperature equations to new time level: $U^{\tau+\Delta\tau}$ $\mu^{\tau+\Delta\tau}$ $\Omega^{\tau+\Delta\tau}$ $\Theta^{\tau+\Delta\tau}$
- Step vertical momentum and geopotential equations (implicit in the vertical): $W^{*\tau+\Delta\tau}$ $\phi^{*\tau+\Delta\tau}$
- Apply implicit Rayleigh damping on W as an adjustment step: $W^{\tau+\Delta\tau} = W^{*\tau+\Delta\tau} - \Delta\tau R_w(\eta) W^{\tau+\Delta\tau}$
- Update final value of geopotential at new time level: $\phi^{\tau+\Delta\tau}$

$$R_w(\eta) = \begin{cases} \gamma_r \sin^2\left[\frac{\pi}{2}\left(1 - \frac{z_{top}-z}{z_d}\right)\right] & \text{for } z \geq (z_{top} - z_d); \\ 0 & \text{otherwise,} \end{cases}$$

$R_w(\eta)$ - damping rate (t^{-1})
z_d - depth of the damping layer
γ_r - damping coefficient

WRF Forecast over Colorado Front Range

Model Initialized 04 Dec 2007 00 UTC

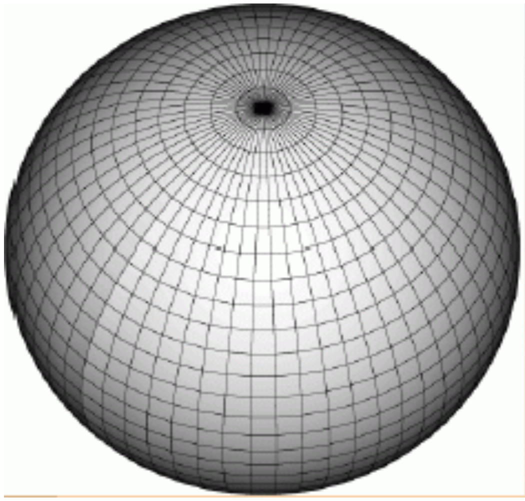


ARW Model: Coordinate Options

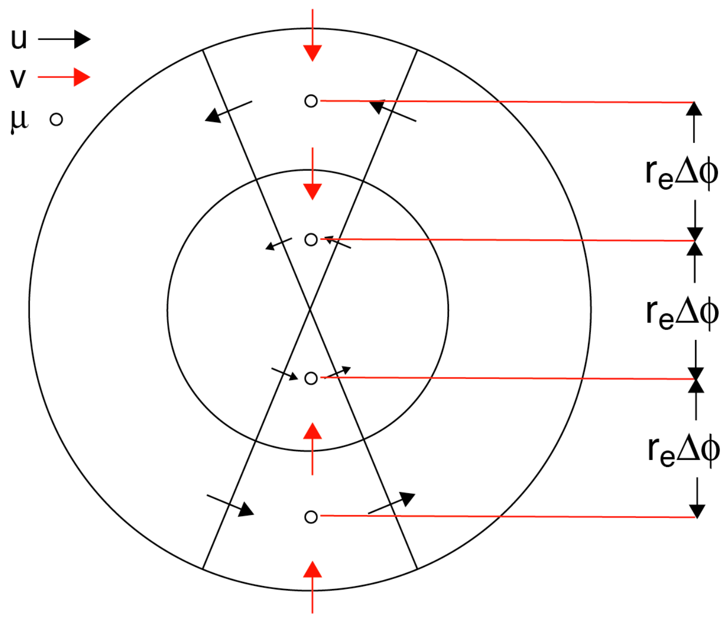
1. Cartesian geometry:
   idealized cases
2. Lambert Conformal:
   mid-latitude applications
3. Polar Stereographic:
   high-latitude applications
4. Mercator:
   low-latitude applications
5. Latitude-Longitude (new in ARW V3)
   global
   regional

Projections 1-4 are isotropic ($m_x = m_y$)
Latitude-Longitude projection is anistropic ($m_x \neq m_y$)

Global ARW - Latitude-Longitude Grid



- Map factors - m_x and m_y
  - Computational grid poles need not be geographic poles.
  - Limited area and nesting capable.
- Polar boundary conditions
- Polar filtering

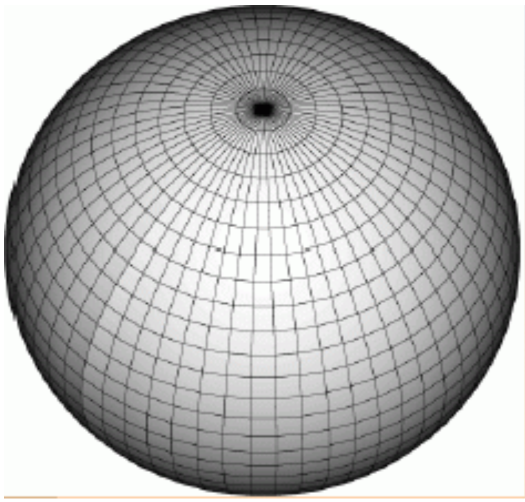


Zero meriodional flux at the poles (cell-face area is zero).

v (poles) only needed for meridional derivative of v near the poles (we interpolate).

All other meriodional derivatives are well-defined near/at poles.

ARW Filters: Polar Filter

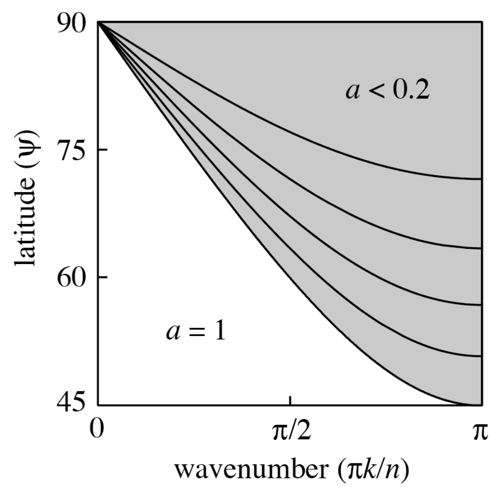


Converging gridlines severely limit timestep.
The polar filter removes this limitation.

Filter procedure - Along a grid latitude circle:

1. Fourier transform variable.
2. Filter Fourier coefficients.
3. Transform back to physical space.

Filter Coefficient $a(k)$, $\psi_o = 45^o$



$$\hat{\phi}(k)_{filtered} = a(k)\,\hat{\phi}(k), \quad \text{for all } k$$

$$a(k) = \min\left[1.,\max\left(0.,\left(\frac{\cos\psi}{\cos\psi_o}\right)^2 \frac{1}{\sin^2(\pi k/n)}\right)\right]$$

$k =$ dimensionless wavenumber
$\hat{\phi}(k) =$ Fourier coefficients from forward transform
$a(k) =$ filter coefficients
$\psi =$ latitude $\psi_o =$ polar filter latitude, filter when $|\psi| > \psi_o$

WRF ARW Model Integration Procedure

Begin time step

Runge-Kutta loop (steps 1, 2, and 3)

(i) advection, p-grad, buoyancy, physics

Acoustic step loop

(i) advance U,V

(ii) advance m, Q

(iii) advance w, f

End acoustic loop

Advance scalars

End Runge-Kutta loop

Adjustment physics (currently microphysics)

End time step

WRF ARW Model Integration Procedure

Begin time step

Runge-Kutta loop (steps 1, 2, and 3)

(i) advection, p-grad, buoyancy, physics

Acoustic step loop

(i) advance U,V (Fourier Filter U,V)

(ii) advance m, Q (Fourier Filter m, Q)

(iii) advance w, f (Fourier Filter w, f)

End acoustic loop

Advance scalars (Fourier Filter Sc)

End Runge-Kutta loop

Adjustment physics (currently microphysics) (Fourier Filter Sc)

End time step

Timestep limited by minimum [illegible]x outside of polar-filter region.
Monotonic and PD transport is not available for global model.

ARW Model: Boundary Condition Options

Lateral boundary conditions

1. Specified (Coarse grid, real-data applications).
2. Open lateral boundaries (gravity-wave radiative).
3. Symmetric lateral boundary condition (free-slip wall).
4. Periodic lateral boundary conditions.
5. Nested boundary conditions (specified).

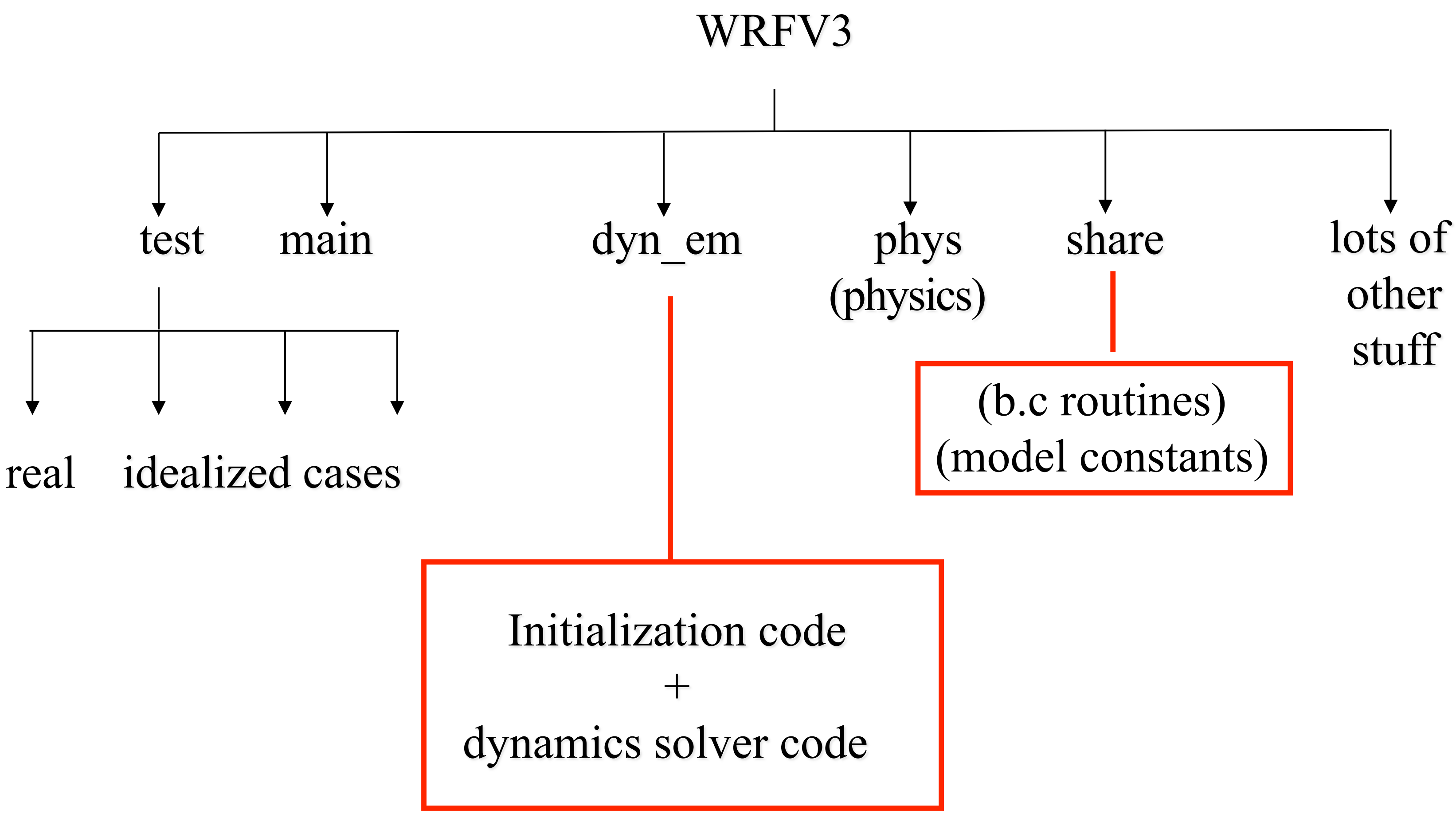
Top boundary conditions

1. Constant pressure.

Bottom boundary conditions

1. Free slip.
2. Various B.L. implementations of surface drag, fluxes.

WRF ARW code



WRF ARW Tech Note

A Description of the Advanced Research WRF Version 3 (June 2008)

http://www.mmm.ucar.edu/wrf/users/pub-doc.html