The Advanced Research WRF (ARW) Dynamics Solver

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WRF Modeling System Flow Chart



WRF ARW Tech Note A Description of the Advanced Research WRF Version 3 http://www.mmm.ucar.edu/wrf/users/pub-doc.html

ARW Dynamical Solver

- Terrain representation
- Vertical coordinate
- Equations / variables
- Time integration scheme
- Grid staggering
- Advection scheme
- Time step parameters
- Filters
- Boundary conditions
- Nesting
- Map projections

Vertical Coordinate and Prognostic Variables



Layer mass $\mu \Delta \eta = \Delta \pi = g \rho \Delta z$ (per unit area)

Conserved state (prognostic) variables:

$$\mu$$
, $U = \mu u$, $V = \mu v$, $W = \mu w$, $\Theta = \mu \theta$

Non-conserved state variable: $\phi = gz$

2D Flux-Form Moist Equations in ARW

Moist Equations:

Moist Equations:

$$\frac{\partial U}{\partial t} + \alpha \mu_{d} \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_{d}} \frac{\partial p}{\partial \eta} \frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g \left(\mu_{d} - \frac{\alpha}{\alpha_{d}} \frac{\partial p}{\partial \eta} \right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \mu_{d}}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial \Omega \theta}{\partial \eta} = \mu Q$$

$$\frac{d\phi}{dt} = gw$$

$$\frac{\partial (\mu_{d}q_{v,l})}{\partial t} + \frac{\partial (Uq_{v,l})}{\partial x} + \frac{\partial (\Omega q_{v,l})}{\partial \eta} = \mu Q_{v,l}$$
Diagnostic relations:

$$\frac{\partial \phi}{\partial \eta} = -\alpha_{d}\mu_{d}, \quad p = \left(\frac{R\Theta}{p_{o}\mu_{d}\alpha_{v}}\right)^{\gamma}$$

Time Integration in ARW

3rd Order Runge-Kutta time integration

advance $\phi^t \rightarrow \phi^{t+\Delta t}$

$$\phi^* = \phi^t + \frac{\Delta t}{3} R(\phi^t)$$
$$\phi^{**} = \phi^t + \frac{\Delta t}{2} R(\phi^*)$$
$$\phi^{t+\Delta t} = \phi^t + \Delta t R(\phi^{**})$$

Amplification factor $\phi_t = ik\phi; \quad \phi^{n+1} = A\phi^n; \quad |A| = 1 - \frac{(k\Delta t)^4}{24}$

Phase and amplitude errors for LF, RK3

Oscillation equation analysis

$$\phi_t = ik\phi$$



Time-Split Runge-Kutta Integration Scheme

$$U_t = L_{fast}(U) + L_{slow}(U)$$



- RK3 is 3rd order accurate for linear eqns, 2nd order accurate for nonlinear eqns.
- Stable for centered and upwind advection schemes.
- Stable for Courant number Udt/dx < 1.73
- Three L_{slow}(U) evaluations per timestep.

WRF ARW Model Integration Procedure

Begin time step



Flux-Form Perturbation Equations

Introduce the perturbation variables:

$$\phi = \overline{\phi}(\overline{z}) + \phi', \ \mu = \overline{\mu}(\overline{z}) + \mu';$$
$$p = \overline{p}(\overline{z}) + p', \ \alpha = \overline{\alpha}(\overline{z}) + \alpha'$$

Note –
$$\phi = \overline{\phi}(\overline{z}) = \overline{\phi}(x, y, \eta),$$

likewise $\overline{p}(x, y, \eta), \overline{\alpha}(x, y, \eta)$

Reduces horizontal pressure-gradient errors.

For small time steps, recast variables as perturbations from time t

$$U' = U'' + U'', V' = V'' + V'', W' = W'' + W'',$$

$$\Theta' = \Theta'' + \Theta'', \mu' = \mu'' + \mu'', \phi' = \phi'' + \phi'';$$

$$p' = p'' + p'', \alpha' = \alpha'' + \alpha''$$

Allows vertical pressure gradient to be expressed in terms of ϕ ".

Small Time Step Integration of Acoustic/Gravity Wave Terms

(Without expanding variables into perturbation form)

$$\begin{split} U^{\tau+\Delta\tau} & \frac{\partial U}{\partial t} + \left(\mu_{d}\alpha\frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_{d}}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x}\right)^{\tau} = R_{U}^{t} \\ \mu_{d}^{\tau+\Delta\tau} & \Omega^{\tau} & \frac{\partial \mu_{d}}{\partial t} + \frac{\partial U}{\partial x}^{\tau+\Delta\tau} + \frac{\partial \Omega}{\partial \eta}^{\tau+\Delta\tau} = 0 \\ \Theta^{\tau+\Delta\tau} & \frac{\partial \Theta}{\partial t} + \left(\frac{\partial U\theta^{t}}{\partial x} + \frac{\partial \Omega\theta^{t}}{\partial \eta}\right)^{\tau+\Delta\tau} = R_{\Theta}^{t} \\ W^{\tau+\Delta\tau} & \begin{cases} \frac{\partial W}{\partial t} + g\overline{\left(\mu_{d} - \frac{\alpha}{\alpha_{d}}\frac{\partial p}{\partial \eta}\right)^{\tau}} = R_{W}^{t} \\ \mu_{d}^{t}\frac{\partial \phi}{\partial t} + U^{\tau+\Delta\tau}\frac{\partial \phi}{\partial x}^{t} + \Omega^{\tau+\Delta\tau}\frac{\partial \phi}{\partial \eta}^{t} - g\overline{W}^{\tau} = R_{\phi}^{t} \end{cases} \end{split}$$

• Forward-backward differencing on U, Θ , and μ equations

• Vertically implicit differencing on W and ϕ equations

Hydrostatic Option

Instead of solving vertically implicit equations for W and ϕ

Integrate the hydrostatic equation to obtain $p(\pi)$:

$$\frac{\partial p}{\partial \eta} = \left(\frac{\alpha_d}{\alpha}\right)^t \mu$$

Recover
$$\alpha$$
 and ϕ from: $p = p_0 \left(\frac{R\theta}{p_0 \alpha_v}\right)^{\gamma}$, and $\frac{\partial \phi}{\partial \eta} = -\mu_d \alpha_d$

W is no longer required during the integration.

ARW model, grid staggering

C-grid staggering



Advection in the ARW Model

2nd, 3rd, 4th, 5th and 6th order centered and upwind-biased schemes are available in the ARW model.

Example: 5th order scheme

$$\frac{\partial (U\psi)}{\partial x} = \frac{1}{\Delta x} \left(F_{i+\frac{1}{2}}(U\psi) - F_{i-\frac{1}{2}}(U\psi) \right)$$

where

$$F_{i-\frac{1}{2}}(U\psi) = U_{i-\frac{1}{2}}\left\{\frac{37}{60}(\psi_{i}+\psi_{i-1}) - \frac{2}{15}(\psi_{i+1}+\psi_{i-2}) + \frac{1}{60}(\psi_{i+2}+\psi_{i-3})\right\}$$
$$-sign(1,U)\frac{1}{60}\left\{(\psi_{i+2}-\psi_{i-3}) - 5(\psi_{i+1}-\psi_{i-2}) + 10(\psi_{i}-\psi_{i-1})\right\}$$

Advection in the ARW Model

For constant U, the 5th order flux divergence tendency becomes

$$\Delta t \frac{\delta(U\psi)}{\Delta x} \bigg|_{5th} = \Delta t \frac{\delta(U\psi)}{\Delta x} \bigg|_{6th}$$
$$- \frac{\left|\frac{U\Delta t}{\Delta x}\right| \frac{1}{60} \left(-\psi_{i-3} + 6\psi_{i-2} - 15\psi_{i-1} + 20\psi_i - 15\psi_{i+1} + 6\psi_{i+2} - \psi_{i+3}\right)}{\frac{Cr}{60} \frac{\partial^6 \psi}{\partial x^6} + H.O.T}$$

The odd-ordered flux divergence schemes are equivalent to the next higher ordered (even) flux-divergence scheme plus a dissipation term of the higher even order with a coefficient proportional to the Courant number.

Maximum Courant Number for Advection

 $C_a = U\Delta t / \Delta x$

| Time | Advection Scheme | | | | |
|-----------------------|------------------|-----------------|----------|----------|----------|
| Integration Scheme | 2 nd | 3 rd | 4^{th} | 5^{th} | 6^{th} |
| Leapfrog (y=0.1) | 0.91 | U | 0.66 | U | 0.57 |
| RK2 | U | 0.90 | U | 0.39 | U |
| RK3 | 1.73 | 1.63 | 1.26 | 1.43 | 1.09 |

U = unstable

(Wicker & Skamarock, 2002)



Mass in a control volume is proportional to

 $(\Delta x \Delta \eta)(\mu)^t$

since
$$\mu(x)\Delta\eta = \Delta\pi = -g\rho\Delta z$$

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$ 2D example

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^{t} \right] = \begin{bmatrix} (\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta}) \end{bmatrix} + \begin{bmatrix} (\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2}) \end{bmatrix}$$
Change in mass over a time step mass fluxes through

5 Έ control volume faces

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

 $\mu \Delta \eta \Delta x$

 Δx

• x

Mass conservation equation

$$\Delta t^{-1} (\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^{t} \right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta} \right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2} \right]$$

Horizontal fluxes through the vertical control-volume faces

 $\Delta \eta$

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^{t} \right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta} \right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2} \right]$$
Vertical fluxes through the horizontal control-volume faces
$$\Delta \eta \left\{ \begin{array}{c} \mu \Delta \eta \Delta x \\ \mu \Delta \eta \Delta x \\ \Delta x \end{array} \right\} x$$

The same mass fluxes are used for neighboring grid cells - hence mass is conserved locally and globally.



| Mass in a control volume | $(\Delta x \Delta \eta)(\mu)^t$ |
|--------------------------|---------------------------------------|
| Scalar mass | $(\Delta x \Delta \eta) (\mu \phi)^t$ |

Mass conservation equation:

$$\Delta t^{-1} (\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t \right] = \begin{bmatrix} (\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta}) \end{bmatrix} + \begin{bmatrix} (\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2}) \end{bmatrix}$$

change in mass over a time step

mass fluxes through control volume faces

Scalar mass conservation equation:

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu \phi)^{t+\Delta t} - (\mu \phi)^{t}) \right] = \begin{bmatrix} (\mu u \phi \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \phi \Delta \eta)_{x+\Delta x/2,\eta}) \end{bmatrix} + \begin{bmatrix} (\mu \omega \phi \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \phi \Delta x)_{x,\eta+\Delta \eta/2}) \end{bmatrix}$$
change in tracer mass over a time step tracer mass fluxes through control volume faces

Moisture Transport in ARW



ARW scheme is conservative, but not positive definite nor monotonic. Removal of negative q results in spurious source of q

Positive-Definite/Monotonic Flux Renormalization

Scalar update, last RK3 step

$$(\mu\phi)^{t+\Delta t} = (\mu\phi)^t - \Delta t \sum_{i=1}^n \delta_{x_i}[f_i]$$
 (1)

n

(1) Decompose flux:
$$f_i = f_i^{upwind} + f_i^c$$

(2) Renormalize high-order correction fluxes f_i^c such that solution is positive definite or monotonic: $f_i^c = R(f_i^c)$

(3) Update scalar eqn. (1) using $f_i = f_i^{upwind} + R(f_i^c)$

Skamarock, MWR 2006, 2241-2250

PD/Monotonic Limiters in ARW - 1D Example Top-Hat Advection

1D Top-hat transport Cr = 0.5, 1 revolution, 200 steps



Moisture Transport in ARW: High Precipitation Bias



2005 ARW 4 km Forecasts:



Moisture Transport in ARW: 24 h ETS and BIAS



Positive-definite advection



ARW Model: Dynamics Parameters

3rd order Runge-Kutta time step

Courant number limited, 1D: $C_r = \frac{U\Delta t}{\Delta x} < 1.43$

Generally stable using a timestep approximately twice as large as used in a leapfrog model.

Acoustic time step

2D horizontal Courant number limited: $C_r = \frac{C_s \Delta \tau}{\Delta h} < \frac{1}{\sqrt{2}}$ $\Delta \tau_{sound} = \Delta t_{RK} / (number of acoustic steps)$

Guidelines for time step

 Δt in seconds should be about $6 * \Delta x$ (grid size in kilometers). Larger Δt can be used in smaller-scale dry situations, but *time_step_sound* (default = 4) should increase proportionately if larger Δt is used. WRF Tutorial July 2013

ARW Filters: Divergence Damping

Purpose: filter acoustic modes (3-D divergence, $D = \nabla \cdot \rho \mathbf{V}$ *)*

$$\left\{ \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla p + \ldots = \gamma'_d \nabla D \right\}$$
$$\nabla \cdot \left\{ \begin{array}{c} \end{array} \right\} \quad \rightarrow \quad \frac{\partial D}{\partial t} + \nabla^2 p + \ldots = \gamma'_d \nabla^2 D$$

From the pressure equation: $p_t \simeq c^2 D$

$$\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla [p_{\tau} + \gamma_d (p^{\tau} - p^{\tau - \Delta \tau})] + \ldots = 0$$

 $\gamma_d = 0.1$ recommended (default)

(Illustrated in height coordinates for simplicity)

ARW Filters: Vertically Implicit Off-Centered Acoustic Step

Purpose: damp vertically-propagating acoustic modes

$$\frac{\partial W}{\partial t} + g \overline{\left(\mu_d - \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta}\right)}^{\tau} = \dots$$
$$\frac{\partial \phi}{\partial t} - \frac{g}{\mu_d^t} \overline{W}^{\tau} = \dots$$
$$\overline{(f_{-})}^{\tau} = \frac{1 + \beta}{2} \overline{(f_{-})}^{\tau + \Delta \tau} + \frac{1 - \beta}{2} \overline{(f_{-})}^{\tau}$$

Slightly forward centering the vertical pressure gradient damps 3-D divergence as demonstrated for the divergence damper

 $\beta = 0.1$ recommended (default)

ARW Filters: External Mode Filter

Purpose: filter the external mode Vertically integrated horizontal divergence, $D_h = \int_1^0 (\nabla_\eta \cdot \mu \mathbf{V}_h) d\eta$

$$\left\{ \frac{\partial \mu \mathbf{V}_h}{\partial t} + \ldots = -\gamma_e \nabla_\eta D_h \right\}$$
$$\int_1^0 \nabla_\eta \cdot \left\{ \begin{array}{c} \\ \end{array} \right\} d\eta \quad \to \quad \frac{\partial D_h}{\partial t} + \ldots = \gamma_e \nabla^2 D_h$$

Continuity equation: $\frac{\partial \mu}{\partial t} = -\nabla_{\eta} \cdot \mu \mathbf{V}_{h} - \frac{\partial \mu \dot{\eta}}{\partial \eta} = D_{h}$

$$\left|\frac{\partial \mu \mathbf{V}_{h}}{\partial \tau} + \ldots = -\gamma_{e} \frac{\Delta x^{2}}{\Delta \tau^{2}} \nabla_{\eta} (\mu^{\tau} - \mu^{\tau - \Delta \tau})\right|$$

 $\gamma_e = 0.01$ recommended (default)

(Primarily for real-data applications)

ARW Filters: Vertical Velocity Damping

Purpose: damp anomalously-large vertical velocities (usually associated with anomalous physics tendencies)

Additional term:

$$\partial_t W = \dots - \mu_d \operatorname{sign}(W) \gamma_w (Cr - Cr_\beta)$$

$$Cr = \left|\frac{\Omega dt}{\mu d\eta}\right|$$

$$Cr_{\beta} = 1.0$$
 typical value (default)
 $\gamma_{w} = 0.3$ m/s² recommended (default)

ARW Filters: 2nd-Order Horizontal Mixing, Horizontal-Deformation-Based K_h

Purpose: mixing on horizontal coordinate surfaces (real-data applications)

$$K_h = C_s^2 l^2 \left[0.25(D_{11} - D_{22})^2 + \overline{D_{12}^2}^{xy} \right]^{\frac{1}{2}}$$

where

$$l = (\Delta x \Delta y)^{1/2}$$

$$D_{11} = 2 m^2 [\partial_x (m^{-1}u) - z_x \partial_z (m^{-1}u)]$$

$$D_{22} = 2 m^2 [\partial_y (m^{-1}v) - z_y \partial_z (m^{-1}v)]$$

$$D_{12} = m^2 [\partial_y (m^{-1}u) - z_y \partial_z (m^{-1}u) + \partial_x (m^{-1}v) - z_x \partial_z (m^{-1}v)]$$

 $C_s = 0.25$ (Smagorinsky coefficient, default value)

Implicit Rayleigh w Damping Layer for Split-Explicit Nonhydrostatic NWP Models (gravity-wave absorbing layer)

Modification to small time step:

- Step horizontal momentum, continuity, and potential temperature equations to new time level:
- Step vertical momentum and geopotential equations (implicit in the vertical):
- Apply implicit Rayleigh damping on *W* as an adjustment step:
- Update final value of geopotential at new time level:

$$\begin{array}{ccc} U^{\tau + \Delta \tau} & \mu^{\tau + \Delta \tau} \\ \Omega^{\tau + \Delta \tau} & \Theta^{\tau + \Delta \tau} \end{array}$$

$$W^{*\tau+\Delta\tau} \quad \phi^{*\tau+\Delta\tau}$$

$$W^{\tau+\Delta\tau} = W^{*\tau+\Delta\tau} - \Delta\tau R_w(\eta)W^{\tau+\Delta\tau}$$

 $\phi^{\tau + \Delta \tau}$

 $R_w(\eta) = \begin{cases} \gamma_r \sin^2 \left[\frac{\pi}{2} \left(1 - \frac{z_{top} - z}{z_d} \right) \right] & \text{for } z \ge (z_{top} - z_d); \\ 0 & \text{otherwise,} \end{cases} \begin{array}{c} R_w(\eta) \text{- damping rate (t^{-1})} \\ z_d \text{- depth of the damping layer} \\ \gamma_r \text{- damping coefficient} \end{cases}$

WRF Forecast over Colorado Front Range

Model Initialized 04 Dec 2007 00 UTC



ARW Model: Coordinate Options

- 1. Cartesian geometry: idealized cases
- 2. Lambert Conformal: mid-latitude applications
- 3. Polar Stereographic: high-latitude applications
- 4. Mercator:
 - low-latitude applications
- 5. Latitude-Longitude (new in ARW V3) global regional

Projections 1-4 are isotropic $(m_x = m_y)$ Latitude-Longitude projection is anistropic $(m_x \neq m_y)$

Global ARW - Latitude-Longitude Grid



- Map factors m_x and m_y
 - Computational grid poles need not be geographic poles.
 - Limited area and nesting capable.
- Polar boundary conditions
- Polar filtering



Zero meriodional flux at the poles (cell-face area is zero).

v (poles) only needed for meridional derivative of v near the poles (we interpolate).

All other meriodional derivatives are well-defined near/at poles.

ARW Filters: Polar Filter



Filter Coefficient a(k), $\psi_0 = 45^\circ$



Converging gridlines severely limit timestep. The polar filter removes this limitation.

Filter procedure - Along a grid latitude circle:

- 1. Fourier transform variable.
- 2. Filter Fourier coefficients.
- 3. Transform back to physical space.

$$\hat{\phi}(k)_{filtered} = a(k) \,\hat{\phi}(k), \quad \text{for all } k$$
$$a(k) = \min\left[1., \max\left(0., \left(\frac{\cos\psi}{\cos\psi_o}\right)^2 \frac{1}{\sin^2(\pi k/n)}\right)\right]$$

k = dimensionless wavenumber $\hat{\phi}(k) = \text{Fourier coefficients from forward transform}$ a(k) = filter coefficients $\psi = \text{latitude } \psi_o = \text{ polar filter latitude, filter when } |\psi| > \psi_o$

WRF ARW Model Integration Procedure

Begin time step



WRF ARW Model Integration Procedure

Begin time step



Timestep limited by minimum Δx outside of polar-filter region. Monotonic and PD transport is not available for global model.

ARW Model: Boundary Condition Options

Lateral boundary conditions

- 1. Specified (Coarse grid, real-data applications).
- 2. Open lateral boundaries (gravity-wave radiative).
- 3. Symmetric lateral boundary condition (free-slip wall).
- 4. Periodic lateral boundary conditions.
- 5. Nested boundary conditions (specified).

Top boundary conditions

1. Constant pressure.

Bottom boundary conditions

- 1. Free slip.
- 2. Various B.L. implementations of surface drag, fluxes.

WRF ARW code



WRF ARW Tech Note

A Description of the Advanced Research WRF Version 3 (June 2008, 2012 update) http://www.mmm.ucar.edu/wrf/users/pub-doc.html