# The Advanced Research WRF (ARW) Dynamics Solver

1. Time integration scheme
2. Time step parameters
3. Advection (transport) and conservation
4. Advection-scheme configurations

**WRF ARW Tech Note**

A Description of the Advanced Research WRF Version 4 (July 2021; WRF Version 4.3)

http://www2.mmm.ucar.edu/wrf/users/docs/technote/contents.html

# 3rd Order Runge-Kutta time integration

$$\frac{\partial U}{\partial t} = RHS_u$$
$$\frac{\partial V}{\partial t} = RHS_v$$
$$\frac{\partial W}{\partial t} = RHS_w$$

•
•
•

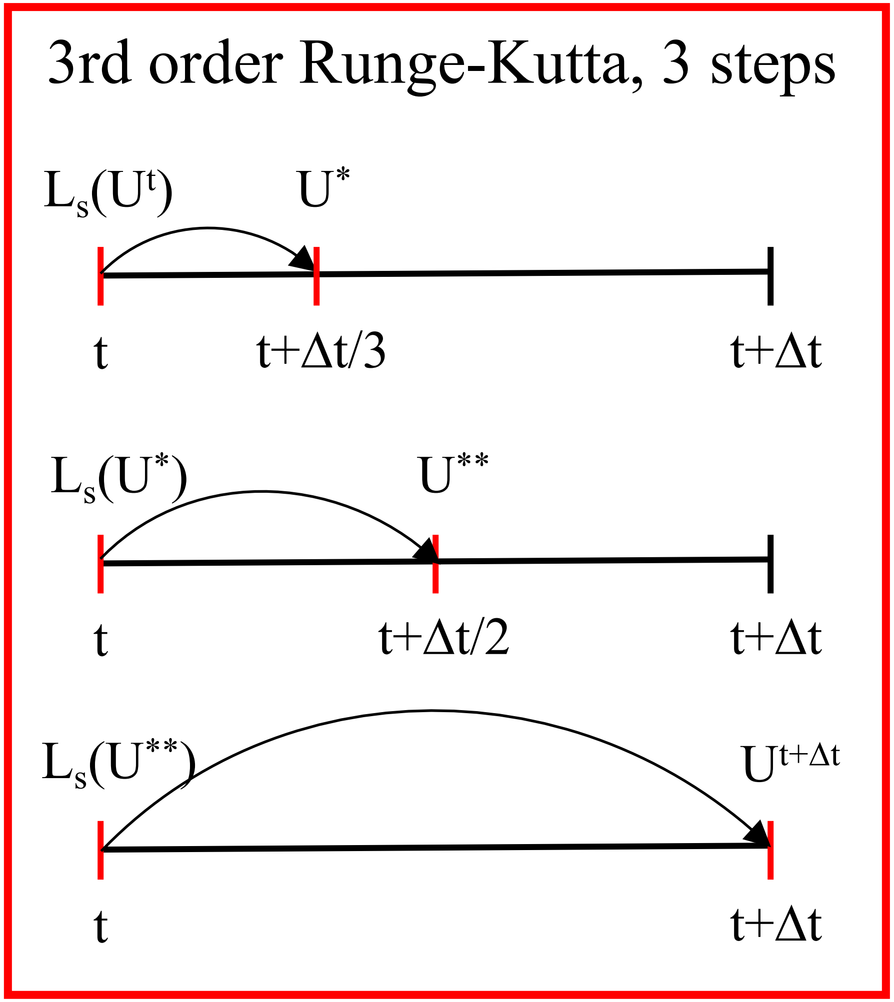
advance $\phi^t \rightarrow \phi^{t+\Delta t}$

$$\phi^* = \phi^t + \frac{\Delta t}{3} RHS(\phi^t)$$
$$\phi^{**} = \phi^t + \frac{\Delta t}{2} RHS(\phi^*)$$
$$\phi^{t+\Delta t} = \phi^t + \Delta t \, RHS(\phi^{**})$$

Amplification factor $\phi_t = i k \phi; \quad \phi^{n+1} = A \phi^n; \quad |A| = 1 - \frac{(k\Delta t)^4}{24}$

# Dynamics: 1. Time integration scheme

$$U_t = L_{slow}(U)$$

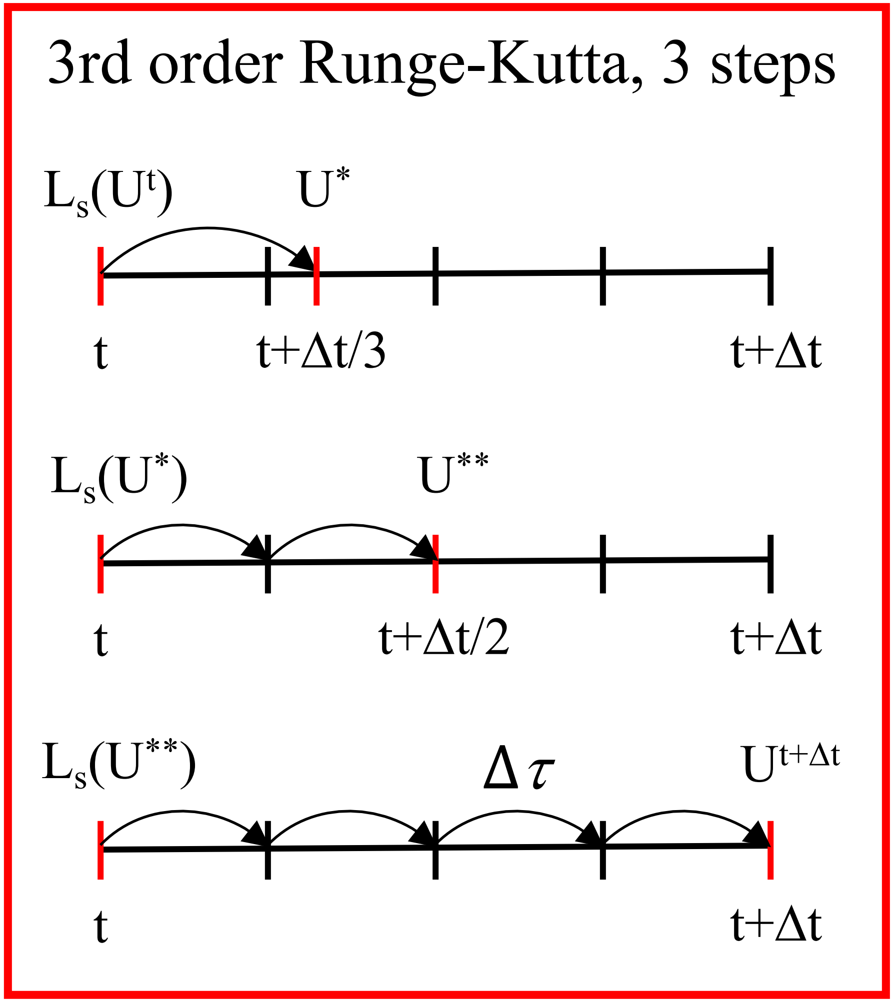



- RK3 is 3rd order accurate for linear eqns, 2nd order accurate for nonlinear eqns.
- Stable for centered and upwind advection schemes.
- Stable for Courant number $U\Delta t/\Delta x < 1.73$
- Three $L_{slow}(U)$ evaluations per timestep.

# Dynamics: 1. Time integration scheme – time splitting

$$U_t = L_{fast}(U) + L_{slow}(U)$$

*fast*: acoustic and gravity wave terms.
*slow*: everything else.

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# Dynamics: 1. Time integration scheme - implementation

Begin time step

Runge-Kutta loop (steps 1, 2, and 3)

(i) advection, p-grad, buoyancy using ($\varphi^{t}$, $\varphi^{*}$, $\varphi^{**}$)

(ii) physics if step 1, save for steps 2 and 3

(iii) mixing, other non-RK dynamics, save…

(iv) assemble dynamics tendencies

Acoustic step loop

(i) advance U,V, then μ, Θ, then w, $\varphi$

(ii) time-average U,V, Ω

End acoustic loop

Advance scalars using time-averaged U,V, Ω

End Runge-Kutta loop

Adjustment physics (currently microphysics)

End time step

3rd order Runge-Kutta time step (the main timestep in WRF) $\Delta t_{RK}$

*Where?* The namelist.input file:

&domains

*time_step (integer seconds)*

*time_step_fract_num*

*time_step_fract_den*

Guidelines for time step selection

$\Delta t_{RK}$ in seconds should be about $6\Delta x$ (grid size in kilometers).

For example, for $\Delta x = 10\ km$ choose $\Delta t = 60$ seconds.

# Dynamics: 2. Time step parameters

3rd order Runge-Kutta time step $\Delta t_{RK}$ (&domains *time_step*)
(the main timestep in WRF)

$\Delta t_{RK}$ in seconds should be about $6\Delta x$ (grid size in kilometers).

## Acoustic time step

2D horizontal Courant number limited:

$$\Delta \tau_{acoustic} = \Delta t_{RK} / \text{(number of acoustic steps)}$$

*Where?* The namelist.input file:

&dynamics

*time_step_sound* (an even integer > = 2)

The ARW default for *time_step_sound* is 4

# Dynamics: 2. Time step parameters

3rd order Runge-Kutta time step $\Delta t_{RK}$ (&domains *time_step*)

Acoustic time step [&dynamics *time_step_sound* (integer)]

Are the RK and/or acoustic timesteps too big?
If ARW blows up (aborts) quickly (possibly a dynamics instability), try:

(1) Decreasing $\Delta t_{RK}$ (this also decreases $\Delta \tau_{acoustic}$), or
(2) increasing the integer time_step_sound
(this decreases $\Delta \tau_{acoustic}$ but does not change $\Delta t_{RK}$).

# Dynamics: 3. Advection (transport) and conservation – dry-air mass

transport | pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial Uu}{\partial x} - \frac{\partial Vu}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha\mu_d\frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial Uv}{\partial x} - \frac{\partial Vv}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha\mu_d\frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial Uw}{\partial x} - \frac{\partial Vw}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U\theta}{\partial x} - \frac{\partial V\theta}{\partial y} - \frac{\partial \Omega\theta}{\partial \eta} + R_\theta + Q_\theta$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

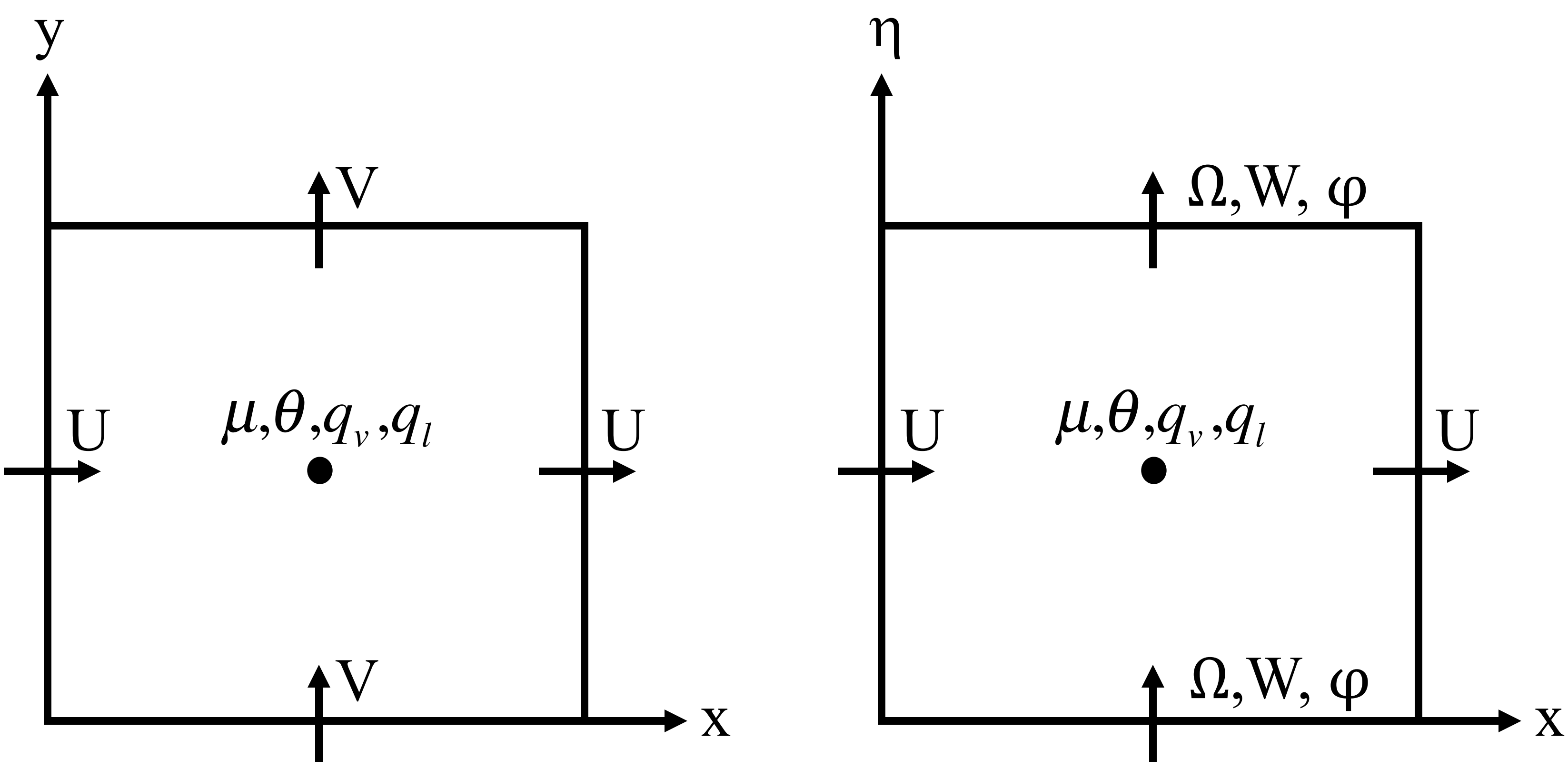
$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

Next:
Dry-air mass conservation in WRF

Diagnostic relations: $\frac{\partial \phi}{\partial \eta} = -\alpha_d\mu_d, p = \left(\frac{R_d\Theta_m}{p_o\mu_d\alpha_d}\right)^\gamma, \Theta_m = \Theta\left(1 + \frac{R_v}{R_d}q_v\right)$

# Dynamics: 3. Grid staggering – horizontal and vertical

C-grid staggering

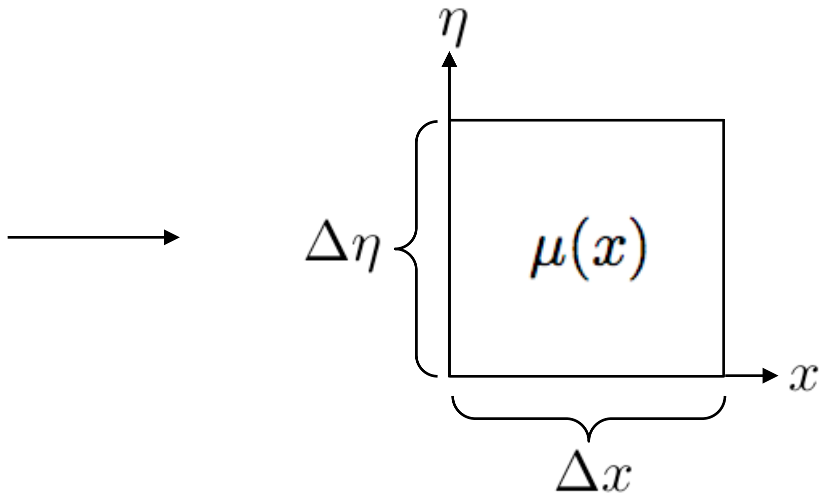



horizontal

vertical

# Dynamics: 3. Advection (transport) and conservation – dry-air mass

control volume
(2D example)




Mass in a control volume is proportional to

$$(\Delta x \Delta \eta)(\mu)^t$$

since $\mu(x)\Delta\eta = \Delta\pi = -g\rho\Delta z$

# Dynamics: 3. Advection (transport) and conservation – dry-air mass

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

2D example

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

Change in mass over a time step

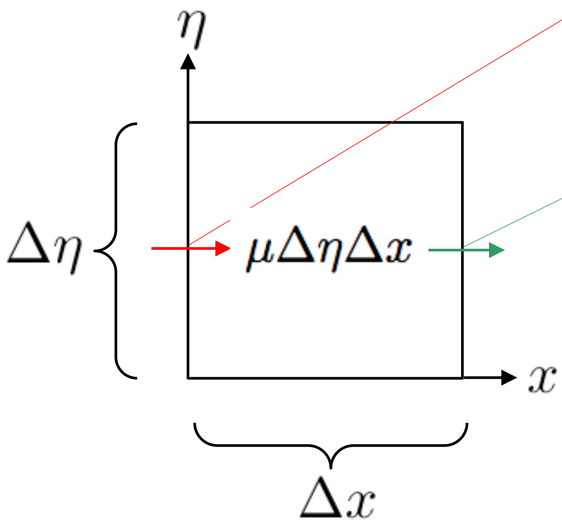
mass fluxes through control volume faces

# Dynamics: 3. Advection (transport) and conservation – dry-air mass

Mass in a control volume $(\Delta x \Delta \eta)(\mu)^t$

Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta}\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$




Horizontal fluxes through the vertical control-volume faces

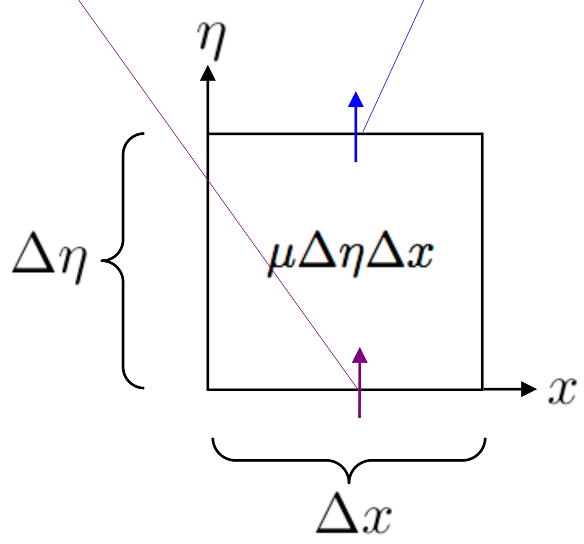
# Dynamics: 3. Advection (transport) and conservation – dry-air mass

Mass in a control volume $(\Delta x \Delta\eta)(\mu)^t$

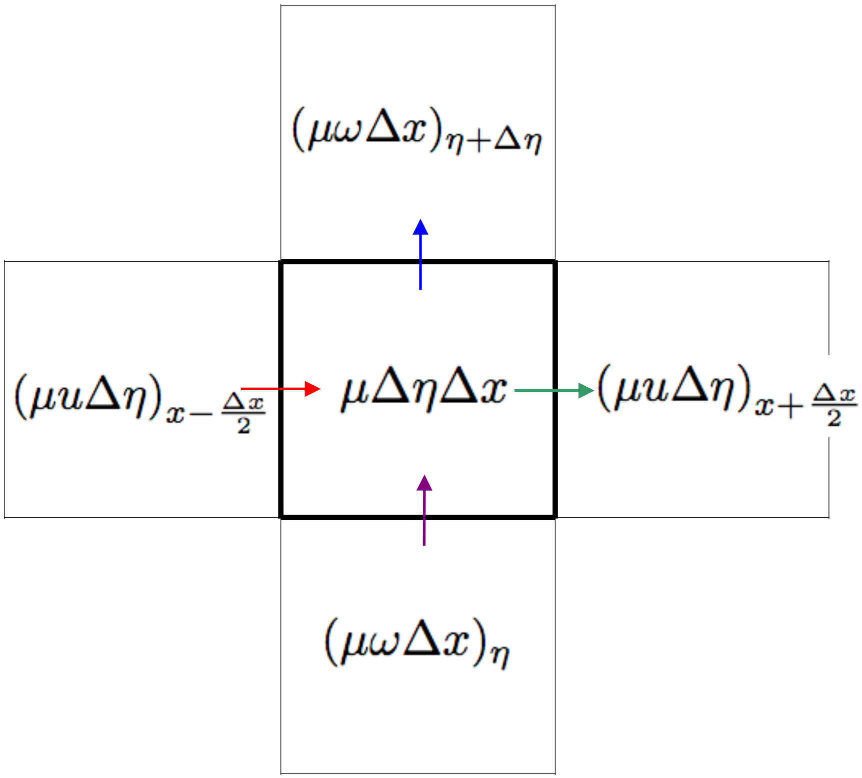
Mass conservation equation

$$\Delta t^{-1}(\Delta x \Delta\eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta\eta)_{x-\Delta x/2,\eta} - (\mu u \Delta\eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu\omega\Delta x)_{x,\eta-\Delta\eta/2} - (\mu\omega\Delta x)_{x,\eta+\Delta\eta/2})\right]$$

Vertical fluxes through the horizontal control-volume faces



The same mass fluxes are used for neighboring grid cells - hence mass is conserved locally and globally.



# Dynamics: 3. Advection (transport) and conservation

transport — pressure gradient

$$\frac{\partial U}{\partial t} = -\frac{\partial U u}{\partial x} - \frac{\partial V u}{\partial y} - \frac{\partial \Omega u}{\partial \eta} - \alpha \mu_d \frac{\partial p}{\partial x} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} + R_u + Q_u$$

$$\frac{\partial V}{\partial t} = -\frac{\partial U v}{\partial x} - \frac{\partial V v}{\partial y} - \frac{\partial \Omega v}{\partial \eta} - \alpha \mu_d \frac{\partial p}{\partial y} - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} + R_v + Q_v$$

$$\frac{\partial W}{\partial t} = -\frac{\partial U w}{\partial x} - \frac{\partial V w}{\partial y} - \frac{\partial \Omega w}{\partial \eta} - g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) + R_w + Q_w$$

$$\frac{\partial \mu_d}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial \Omega}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial U \theta}{\partial x} - \frac{\partial V \theta}{\partial y} - \frac{\partial \Omega \theta}{\partial \eta} + R_\theta + Q_\theta$$

$$\frac{\partial \mu_d q_j}{\partial t} = -\frac{\partial U q_j}{\partial x} - \frac{\partial V q_j}{\partial y} - \frac{\partial \Omega q_j}{\partial \eta} + R_{q_j} + Q_{q_j}$$

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} - \omega\frac{\partial \phi}{\partial \eta} + gw$$

Entropy and scalar mass conservation in WRF

Diagnostic relations: $\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, \; p = \left(\frac{R_d \Theta_m}{p_o \mu_d \alpha_d}\right)^{\gamma}, \; \Theta_m = \Theta\left(1 + \frac{R_v}{R_d} q_v\right)$

# Dynamics: 3. Advection (transport) and conservation – scalars

| | |
|---|---|
| Mass in a control volume | $(\Delta x \Delta \eta)(\mu)^t$ |
| Scalar mass | $(\Delta x \Delta \eta)(\mu\phi)^t$ |

Mass conservation equation:

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu)^{t+\Delta t} - (\mu)^t)\right] = \left[(\mu u \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

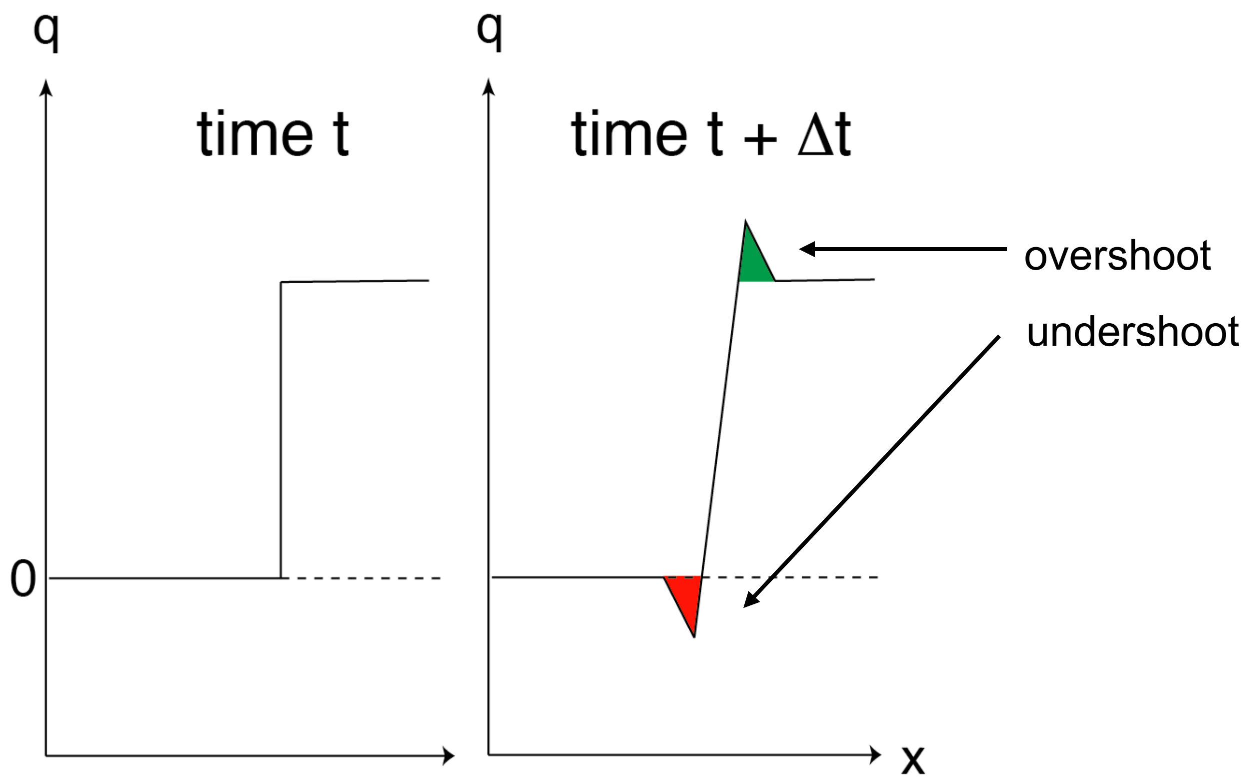
change in mass over a time step — mass fluxes through control volume faces

Scalar mass conservation equation:

$$\Delta t^{-1}(\Delta x \Delta \eta) \cdot \left[(\mu\phi)^{t+\Delta t} - (\mu\phi)^t)\right] = \left[(\mu u \phi \Delta \eta)_{x-\Delta x/2,\eta} - (\mu u \phi \Delta \eta)_{x+\Delta x/2,\eta})\right] + \left[(\mu \omega \phi \Delta x)_{x,\eta-\Delta \eta/2} - (\mu \omega \phi \Delta x)_{x,\eta+\Delta \eta/2})\right]$$

change in tracer mass over a time step — tracer mass fluxes through control volume faces

## 1D advection




ARW transport is conservative,
but not positive definite nor shape preserving.
Removal of negative q
results in spurious source of q .

# Dynamics: 3. Advection (transport) and conservation – shape preserving

Scalar update, last RK3 step

$$(\mu\phi)^{t+\Delta t} = (\mu\phi)^t - \Delta t \sum_{i=1}^{n} \delta_{x_i}[f_i] \qquad (1)$$

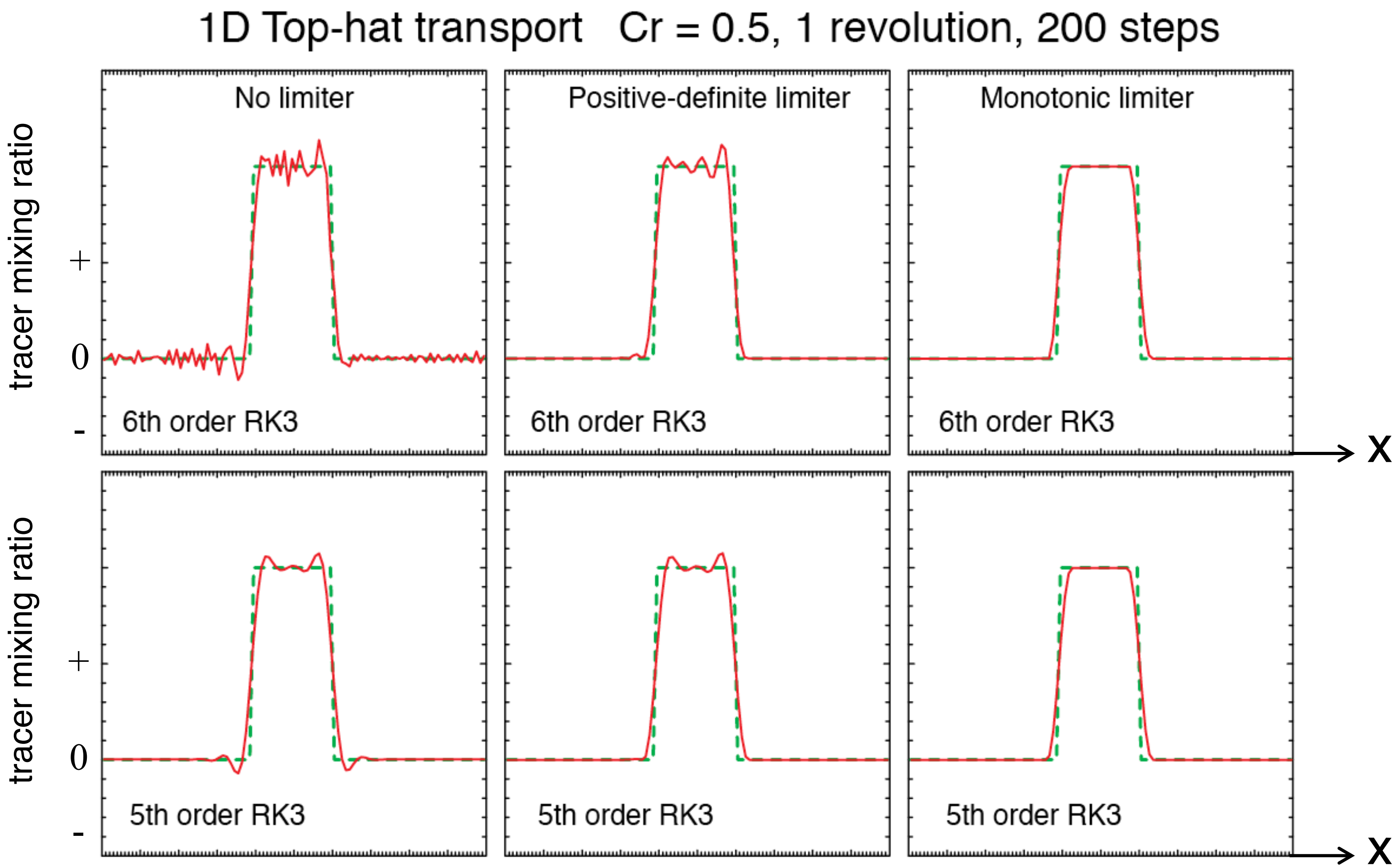
(1) Decompose flux: $f_i = f_i^{upwind} + f_i^c$

(2) Renormalize high-order correction fluxes $f_i^c$ such that solution is positive definite or monotonic: $f_i^c = R(f_i^c)$

(3) Update scalar eqn. (1) using $f_i = f_i^{upwind} + R(f_i^c)$

This is a form of flux-correct transport (Zalesak 1979)

# 1D Example: Top-Hat Advection



# Dynamics: 4. Advection scheme configurations

*Where are the transport-scheme parameters?*

The namelist.input file:

&dynamics

*h_mom_adv_order*
*v_mom_adv_order*
*h_sca_adv_order*
*v_sca_adv_order*

→ scheme order (2, 3, 4, 5 or 6)
defaults:
horizontal (h_*) = 5
vertical (v_*) = 3

*momentum_adv_opt* →

= 1 standard scheme
= 3 5th order WENO
default: 1

*moist_adv_opt*
*scalar_adv_opt*
*chem_adv_opt*
*tracer_adv_opt*
*tke_adv_opt*

→ options:
= 1, 2, 3 : no limiter, positive definite (PD), monotonic
= 4 : 5th order WENO
= 5 : 5th order PD WENO

# Dynamics: 4. Advection scheme configurations

*Where are the transport-scheme parameters?*

The namelist.input file:
&dynamics

- The positive definite limiter (option 2) is enabled in the default WRF configuration.
- Chemistry applications typically use the monotonic limiter (option 3).
- Option 5 (PD WENO) is used in some applications employing multi-moment microphysics.

scheme order (2, 3, 4, 5 or 6)
defaults:
horizontal (h_*) = 5
vertical (v_*) = 3

= 1 standard scheme
= 3 5th order WENO
default: 1

*moist_adv_opt*
*scalar_adv_opt*
*chem_adv_opt*
*tracer_adv_opt*
*tke_adv_opt*

options:
= 1, 2, 3 : no limiter,
positive definite (PD),
monotonic
= 4 : 5th order WENO
= 5 : 5th order PD WENO

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